# Playing Muller Games in a Hurry *

John Fearnley[1] and Martin Zimmermann[2]

[1] Department of Computer Science University of Warwick, UK
john@dcs.warwick.ac.uk
[2] Lehrstuhl Informatik 7, RWTH Aachen University, Germany
zimmermann@automata.rwth-aachen.de

**Motivation.** Many winning conditions for infinite graph-based games depend on the vertices that are visited infinitely often, i.e., the winner of a play cannot be determined after any finite number of steps. We are interested in the following question: is it nevertheless possible to give a criterion to define an "equivalent" finite duration variant of an infinite game? Such a criterion has to stop a play after a finite number of steps and then declare a winner based on the finite play constructed thus far. Such a criterion is sound if Player $i$ has a winning strategy for the infinite duration game if and only if she has a winning strategy for the finite duration game.

McNaughton considered this question from a different perspective. His motivation was to make infinite games suitable for "casual living room recreation" [2]. As human players cannot play infinitely long, he envisions a referee that stops a play at a certain time and declares a winner.

We pursue theoretical questions arising from this idea. If there exists a sound criterion to stop a play after at most $n$ steps, this yields a simple algorithm to determine the winner of the infinite game: the finite duration game can be seen as a reachability game on a finite tree of depth at most $n$, for which simple and efficient algorithms exist. Furthermore, a positive answer to the question whether a winning strategy for the reachability game can be turned into a (small finite-state) winning strategy should yield better results in the average case (although not in the worst case) than standard game reductions.

Consider the following criterion: the players move the token through the arena until a vertex is visited for the second time. Then, the winner of the induced infinite play is declared to be the winner of the finite play. If the game is determined with positional strategies, then this criterion is sound.

We consider Muller games (as McNaughton did), which are in general not positionally determined. Here, the first loop of a play is typically not an indicator of how the infinite play evolves, as the memory allows a player to make different decisions when a vertex is seen again. The criterion for positionally determined games can easily be extended to games that are determined with finite-state strategies by fixing a suitable memory structure and waiting for a repetition of a memory state. However, this bounds the maximal play length only by the size of the memory structure, which is of factorial size for Muller games.

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**Finite-time Muller Games.** The *score* for every subset $F$ of the set of vertices $V$, denoted by $\mathrm{Sc}_F\colon V^+ \to \mathbb{N}$, is defined as

$$\mathrm{Sc}_F(w) = \max\{k \in \mathbb{N} \mid \exists x_1, \ldots, x_k \in V^+ \text{ such that } \mathrm{Occ}(x_i) = F \text{ for all } i \text{ and } x_1 \cdots x_k \text{ is a suffix of } w\} ,$$

where $\mathrm{Occ}(x_i)$ contains the vertices appearing in $x_i$. Also, for every $\mathcal{F} \subseteq 2^V$ we define $\mathrm{MaxSc}_{\mathcal{F}}\colon V^+ \cup V^\omega \to \mathbb{N} \cup \{\infty\}$ by $\mathrm{MaxSc}_{\mathcal{F}}(\rho) = \max_{F \in \mathcal{F}} \max_{w \sqsubseteq \rho} \mathrm{Sc}_F(w)$. To be able to declare a winner of a finite play based on scores, we show that draws cannot occur: (1) *If you play long enough, some score value will be high:* every $w \in V^*$ with $|w| \geq k^{|V|}$ satisfies $\mathrm{MaxSc}_{2^V}(w)$. This also bounds the maximal play length. For every $k > 0$ there is a word $w_k$ of length $k^{|V|} - 1$ such that $\mathrm{MaxSc}_{2^V}(w) < k$, i.e., the bound is tight. (2) *No two scores increase at the same time:* let $k, l \geq 2$, let $F, F' \subseteq V$, let $w \in V^*$ and $v \in V$ with $\mathrm{Sc}_F(w) < k$ and $\mathrm{Sc}_{F'}(w) < l$. If $\mathrm{Sc}_F(wv) = k$ and $\mathrm{Sc}_{F'}(wv) = l$, then $F = F'$.

A finite-time Muller game$(G, \mathcal{F}_0, \mathcal{F}_1, k)$ consists of an arena $G$ with vertex set $V$, a partition $(\mathcal{F}_0, \mathcal{F}_1)$ of $2^V$, and a threshold $k \geq 2$. A play is a finite path $w = w_0 \cdots w_n$ with $\mathrm{MaxSc}_{2^V}(w_0 \cdots w_n) = k$, but $\mathrm{MaxSc}_{2^V}(w_0 \cdots w_{n-1}) < k$, i.e, play is stopped as soon as the threshold score is reached for the first time. Then, there is a unique $F \subseteq V$ such that $\mathrm{Sc}_F(w) = k$. Player 0 wins the play $w$ if $F \in \mathcal{F}_0$ and Player 1 wins otherwise.

In [2], McNaughton considered a slightly different definition: rather than stopping the play when the score of a set reaches the global threshold $k$, his version stops the play when the score of a set $F$ reaches $|F|! + 1$. He proved that the winning regions of a Muller game and his version of a finite-time Muller game coincide. Our main theorem improves the threshold $|F|! + 1$ to the constant 3:

**Theorem 1.** *The winning regions in a Muller game $(G, \mathcal{F}_0, \mathcal{F}_1)$ and in the finite-time Muller game $(G, \mathcal{F}_0, \mathcal{F}_1, 3)$ coincide.*

This theorem is a consequence of a stronger statement about winning strategies in Muller games: in their winning region (denoted by $W_0, W_1$), both players can prevent their opponent from reaching a score of 3. This fact and determinacy of Muller games suffice to prove Theorem 1.

**Lemma 2.** *Player $i$ has a winning strategy $\sigma$ in a Muller game $\mathcal{G} = (G, \mathcal{F}_0, \mathcal{F}_1)$ such that $\mathrm{MaxSc}_{\mathcal{F}_{1-i}}(\mathrm{Play}(v, \sigma, \tau)) \leq 2$ for every $v \in W_i$ and every $\tau \in \Pi_{1-i}$.*

There are Muller games in which the winning player cannot prevent her opponent from reaching score 2, which shows that the bound 2 in Lemma 2 is optimal. However, it is open whether the finite-time Muller game with threshold 2 is always equivalent to the corresponding Muller game.

## References


1. John Fearnley and Martin Zimmermann. Playing Muller games in a hurry. In Angelo Montanari, Margherita Napoli, and Mimmo Parente, editors, *GandALF*, volume 25 of *EPTCS*, pages 146–161, 2010.
2. Robert McNaughton. Playing infinite games in finite time. In Arto Salomaa, Derick Wood, and Sheng Yu, editors, *A Half-Century of Automata Theory*, pages 73–91. World Scientific, 2000.