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Adding Monotonic Counters to Automata and Transition Graphs

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RHEINISCH-WESTFÄLISCHE TECHNISCHE HOCHSCHULE AACHEN

Motivations: Parikh Automata (Klaedtke & Rueß, ICALP 2003)

Idea: how to recognize a language with *arithmetical properties*, such as $L_{\mathrm{abc}} := \{a^k b^k c^k \mid k \geq 1\}$?

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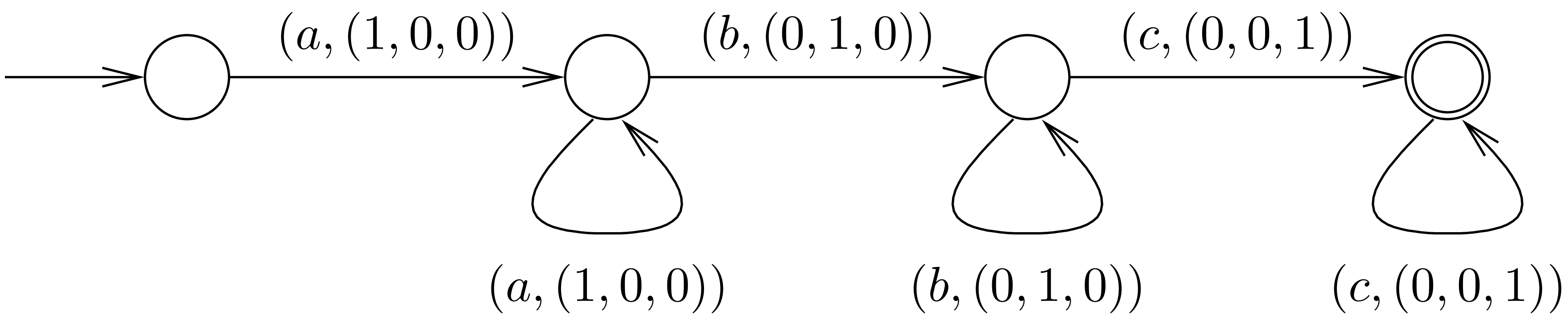
Idea: how to recognize a language with *arithmetical properties*, such as $L_{\mathrm{abc}} := \{a^k b^k c^k \mid k \geq 1\}$?

- Use a finite automaton recognizing $a^+ b^+ c^+$.
- Assign a *vector* to each input symbol.
- Put a Presburger *constraint* on the summed vector (x, y, z): $x = y = z$.



Outline

1. Parikh automata
   - Semi-linear sets and Parikh's theorem
   - Parikh automata and the Chomsky hierarchy
2. Monotonic-counter extensions of (infinite) graphs
   - Some classes of infinite graphs
   - Monotonic-counter extensions
3. Reachability problem

Part 1

Parikh Automata

$A \subseteq \mathbb{N}^n$ *linear*: $A = \{\bar{x}_0 + k_1\bar{x}_1 + \ldots + k_m\bar{x}_m \mid k_1, \ldots, k_m \in \mathbb{N}\}$
for some $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m \in \mathbb{N}^n$

Semi-linear set: finite union of linear sets.

Example: $B := \{(x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_1 < x_2 < x_3\}$ is linear:

$$\{(0,1,2) + k_1(0,0,1) + k_2(0,1,1) + k_3(1,1,1) \mid k_1, k_2, k_3 \in \mathbb{N}\}.$$

Semi-Linear Sets and Parikh's Theorem

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Properties of semi-linear sets:

- effective closure under Boolean operations [Ginsburg & Spanier]
- equivalence to Presburger-definable sets [Ginsburg & Spanier]

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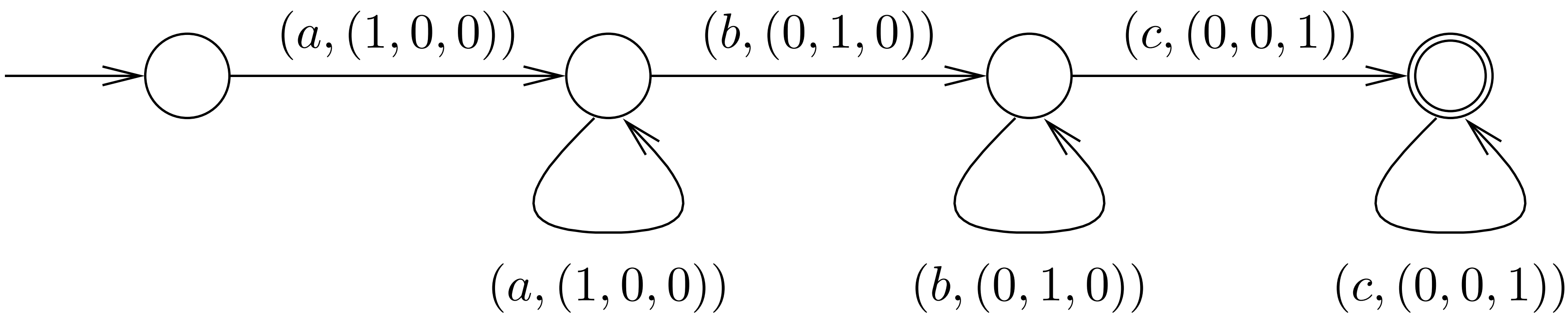
Parikh's theorem: The *Parikh image* of any context-free language is effectively semi-linear.

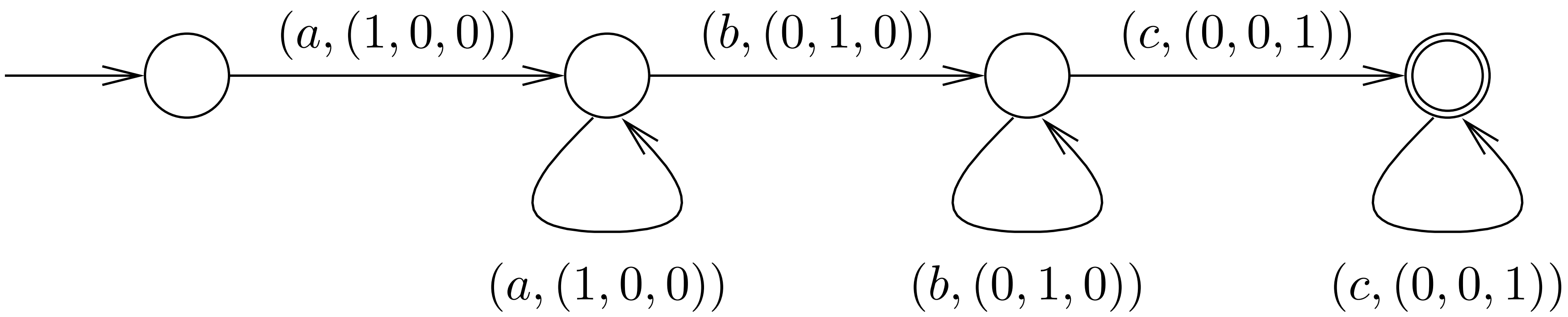
Parikh Automata



Parikh finite automaton (*Parikh-FA*) $(\mathfrak{A}, C)$ of dimension $n \geq 1$ over Σ:

- finite automaton $\mathfrak{A}$ over $\Sigma \times D$ ($D \subseteq \mathbb{N}^n$ finite, nonempty)
- semi-linear set $C \subseteq \mathbb{N}^n$

Parikh Automata



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Word $u := a_1 \cdots a_m$ is *accepted* iff

- $v := (a_1, \bar{d}_1) \cdots (a_m, \bar{d}_m) \in L(\mathfrak{A})$ exists, for some $\bar{d}_1, \ldots, \bar{d}_m \in D$,
- *and* $\Phi(v) := \bar{d}_1 + \cdots + \bar{d}_m \in C$.

extended Parikh mapping $\Phi\colon (\Sigma \times D)^* \to \mathbb{N}^n$

Emptiness Problem

Lemma (Klaedtke & Rueß):

If the *Parikh image* of $L \subseteq (\Sigma \times D)^*$ is effectively semi-linear, then also its *extended Parikh image* $\Phi(L)$.

Theorem (Klaedtke & Rueß):

The emptiness problem for Parikh-FA's is decidable.

Proof idea.

$$L(\mathfrak{A}, C) \neq \emptyset \qquad \text{iff} \qquad \underbrace{\Phi(L(\mathfrak{A})) \cap C \neq \emptyset}$$

– Both sets are semi-linear.

– Intersection of semi-linear sets is effectively semi-linear.

$\Longrightarrow$ (Non-)Emptiness is decidable.

□

Parikh Automata and the Chomsky Hierarchy

Automata of the Chomsky hierarchy as the automaton component $\mathfrak{A}$:

Turing machines

linear-bounded automata

pushdown automata

finite automata

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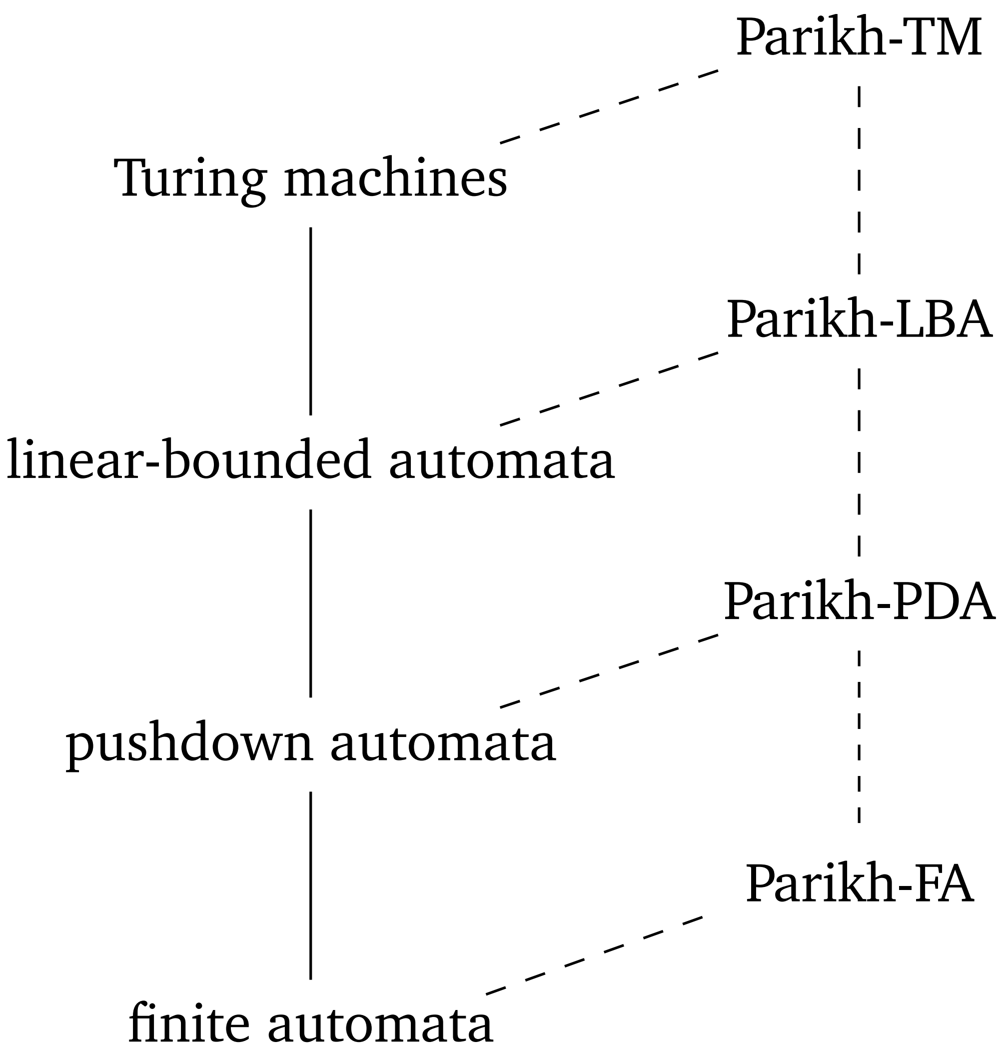
Automata of the Chomsky hierarchy as the automaton component $\mathfrak{A}$:



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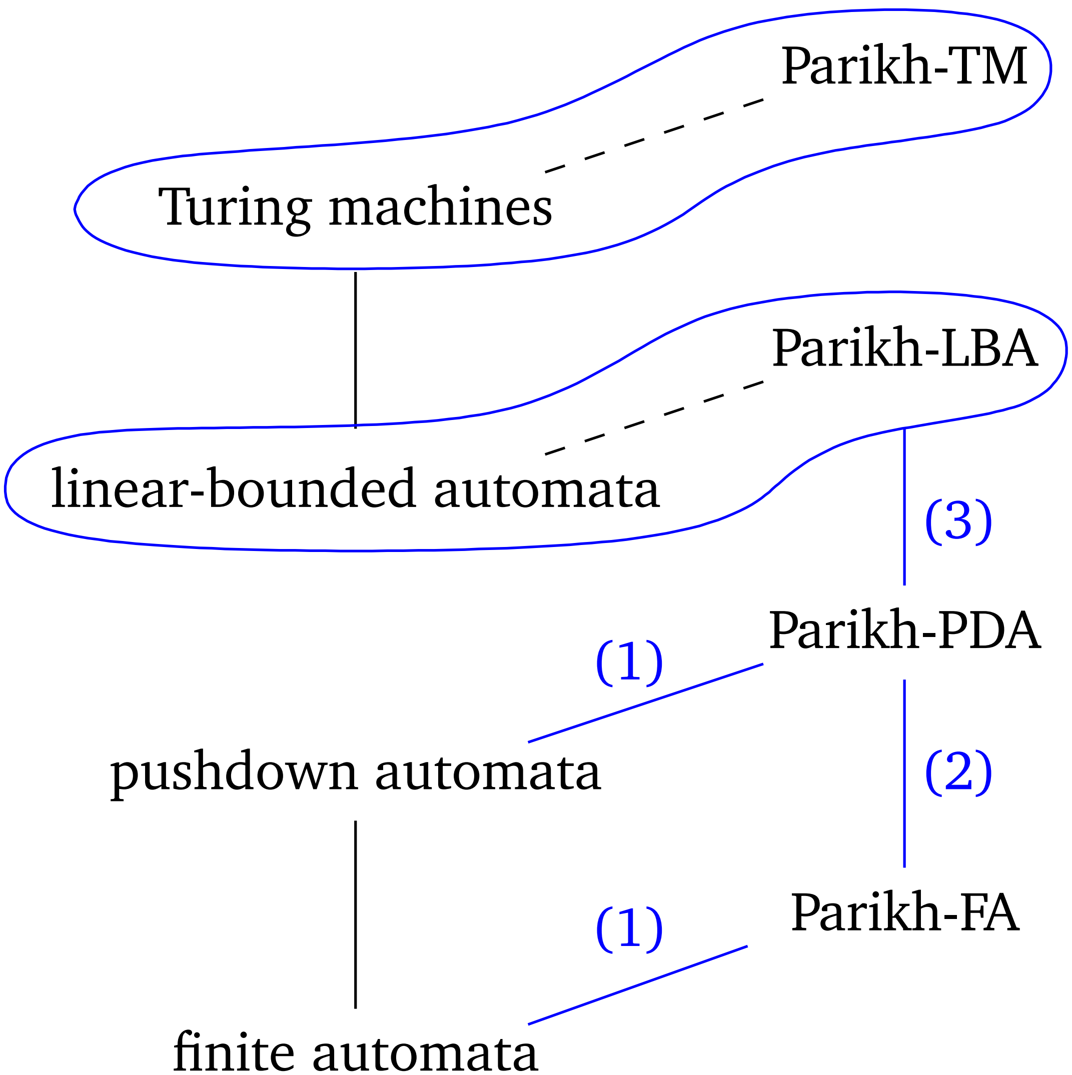


1. $\{a^k b^k c^k \mid k \geq 1\}$
2. $\{ww^{\mathrm{R}} \mid w \in \{a, b\}^*\}$
3. semi-linearity of Parikh-PDA recognizable languages

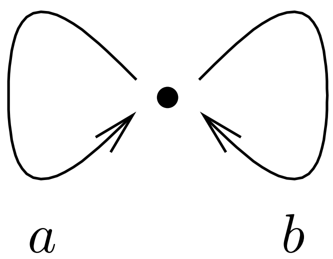
Part 2

Monotonic-Counter Extensions of (Infinite) Graphs

Monotonic-Counter Graphs

Σ-labeled graph

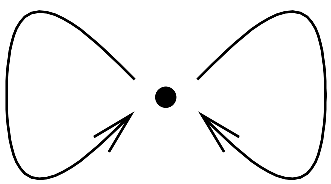
$G := (V, (E_a)_{a \in \Sigma})$



$(\Sigma \times D)$-labeled graph

$G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D})$

($D \subseteq \mathbb{N}^n$ finite, nonempty)



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$\rightsquigarrow$

Monotonic-counter extension of G:

Σ-labeled graph $\widetilde{G} := (\widetilde{V}, (\widetilde{E}_a)_{a \in \Sigma})$ with $\widetilde{V} := V \times \mathbb{N}^n$ and $((\alpha, \bar{x}), (\beta, \bar{y})) \in \widetilde{E}_a$ iff

$(\alpha, \beta) \in E_{(a,\bar{d})}$ and

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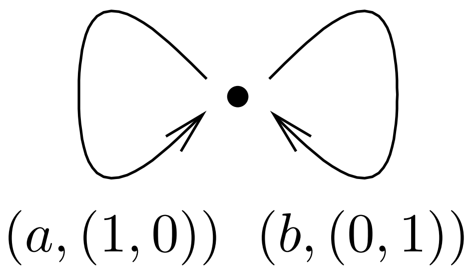
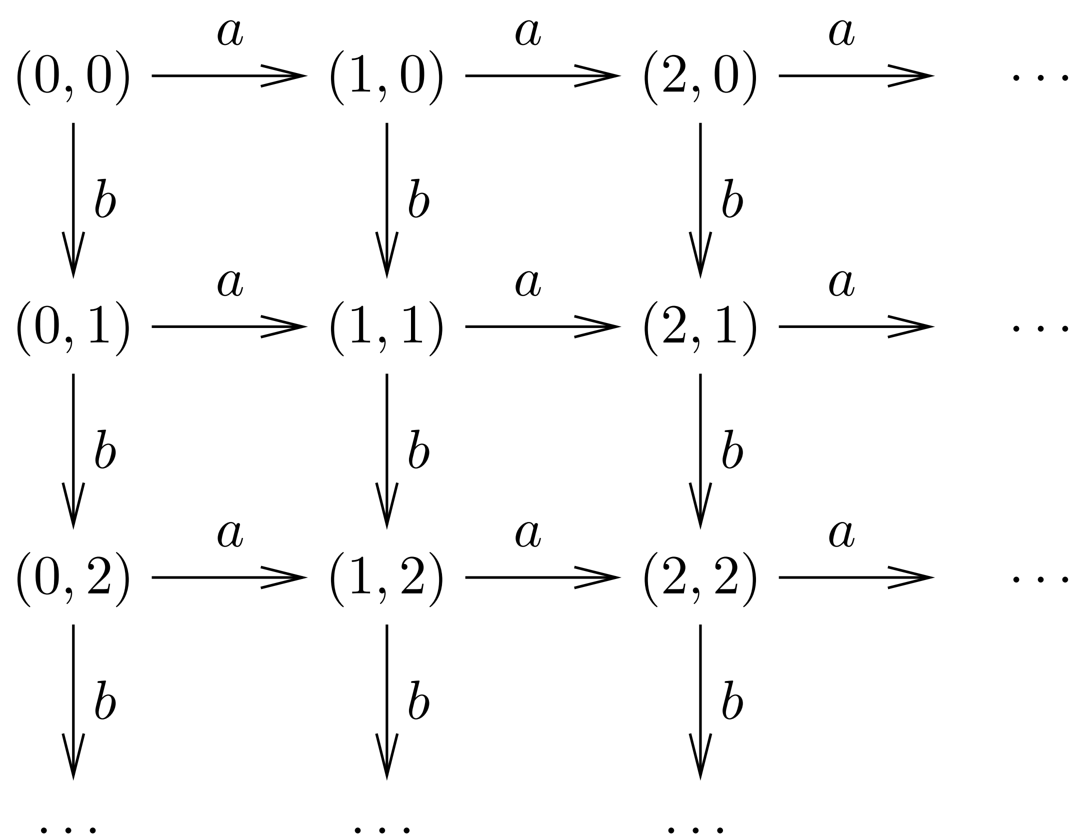




Some Classes of Infinite Graphs with Finite Representations

Vertices: regular sets over alphabet Γ

Edges: automaton-definable relations over words, e.g.:

- *pushdown graphs* [Muller & Schupp]: transitions of ε-free pushdown automata
- *prefix-recognizable graphs* [Caucal]: generalized prefix rewriting rules
- *synchronized rational graphs* or *automatic graphs* [Frougny & Sakarovitch, Blumensath & Grädel]: synchronized rational relations
- *rational graphs* [Morvan]: rational relations

Hierarchy of Graph Classes

rational graphs

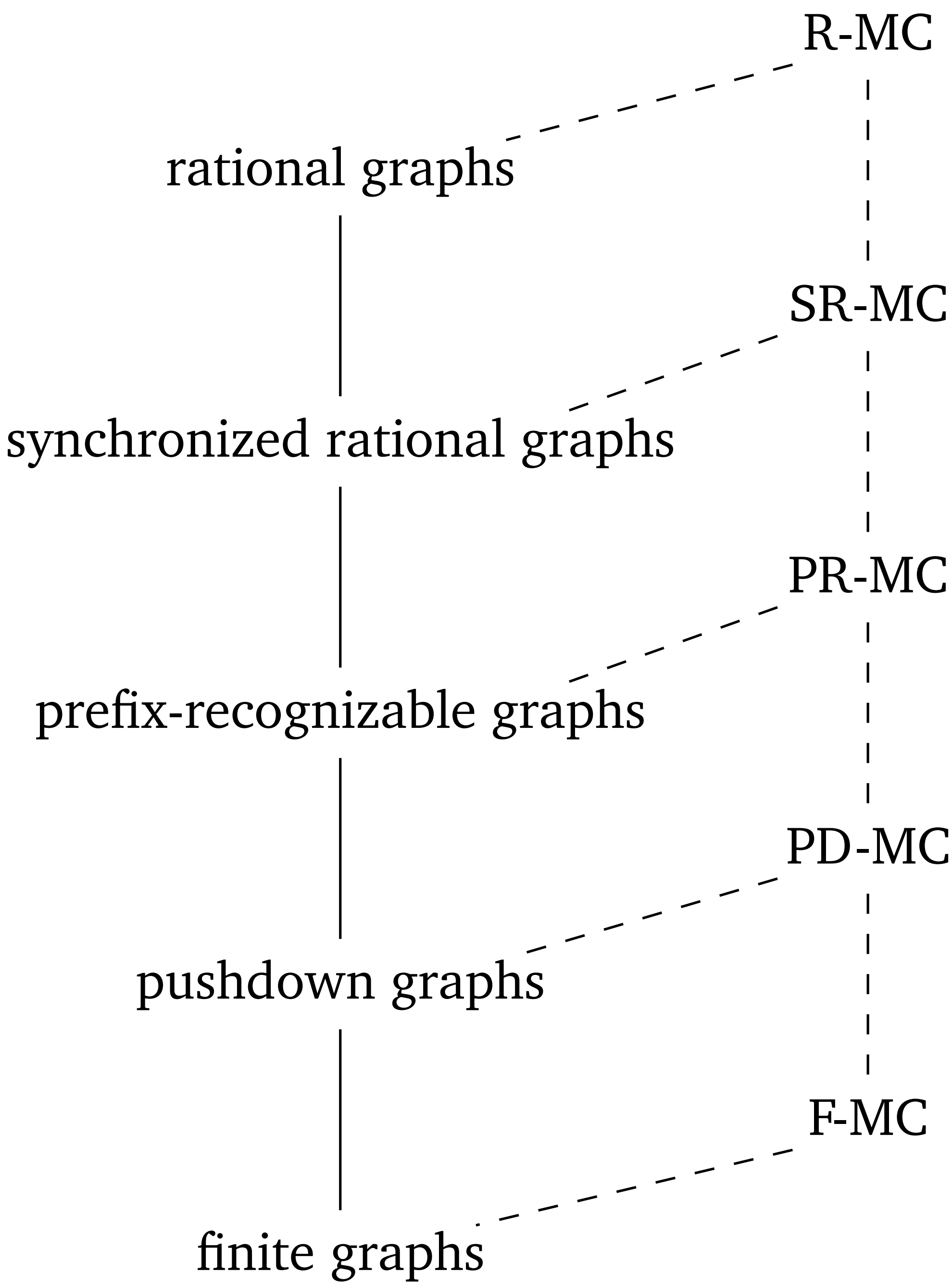
synchronized rational graphs

prefix-recognizable graphs

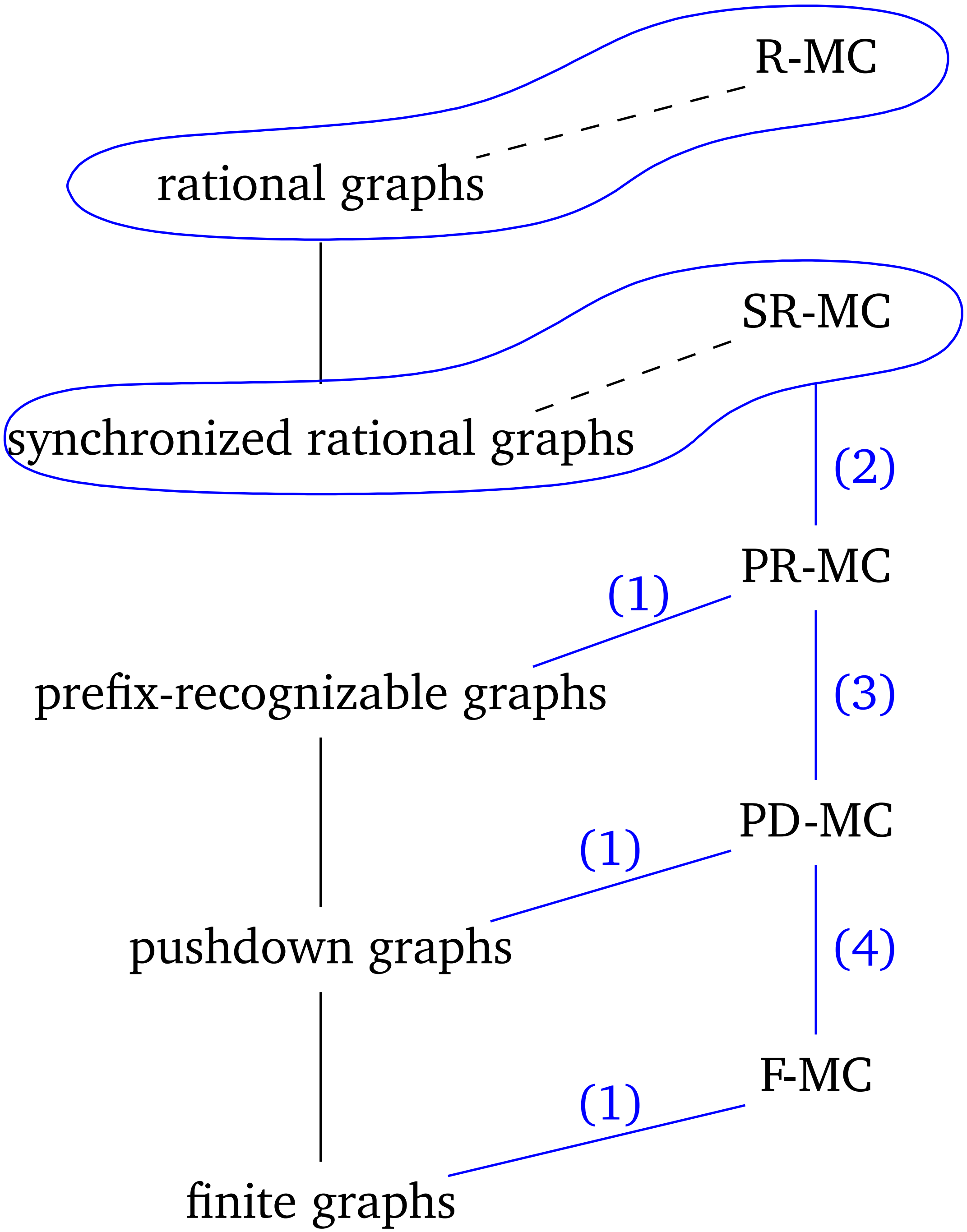
pushdown graphs

finite graphs

Hierarchy of Graph Classes



Hierarchy of Graph Classes

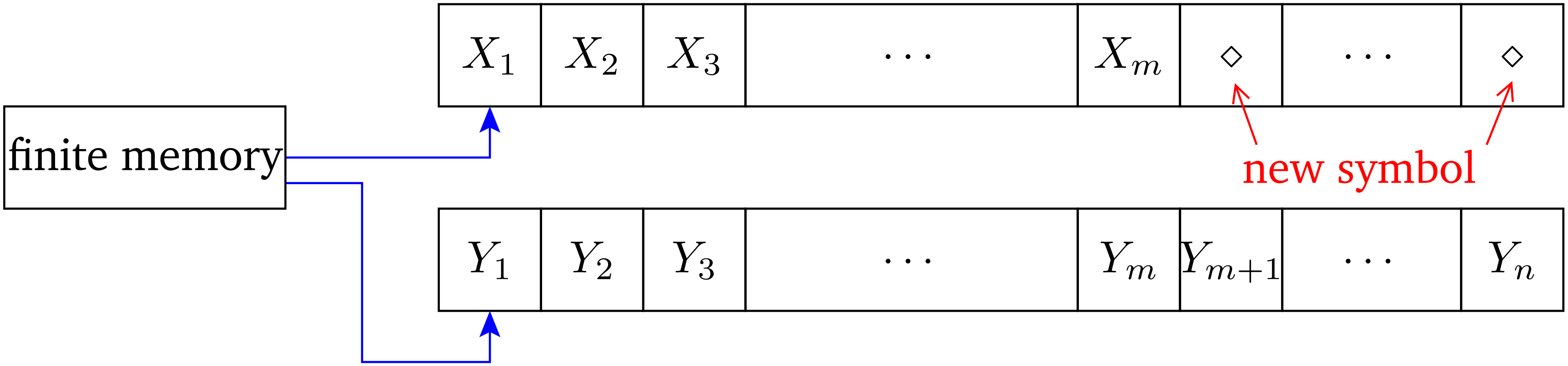


1. infinite two-dimensional grid
2. decidability of the reachability problem
3. graphs with vertices of unbounded degree
4. graphs with repetition-free cycles of unbounded length

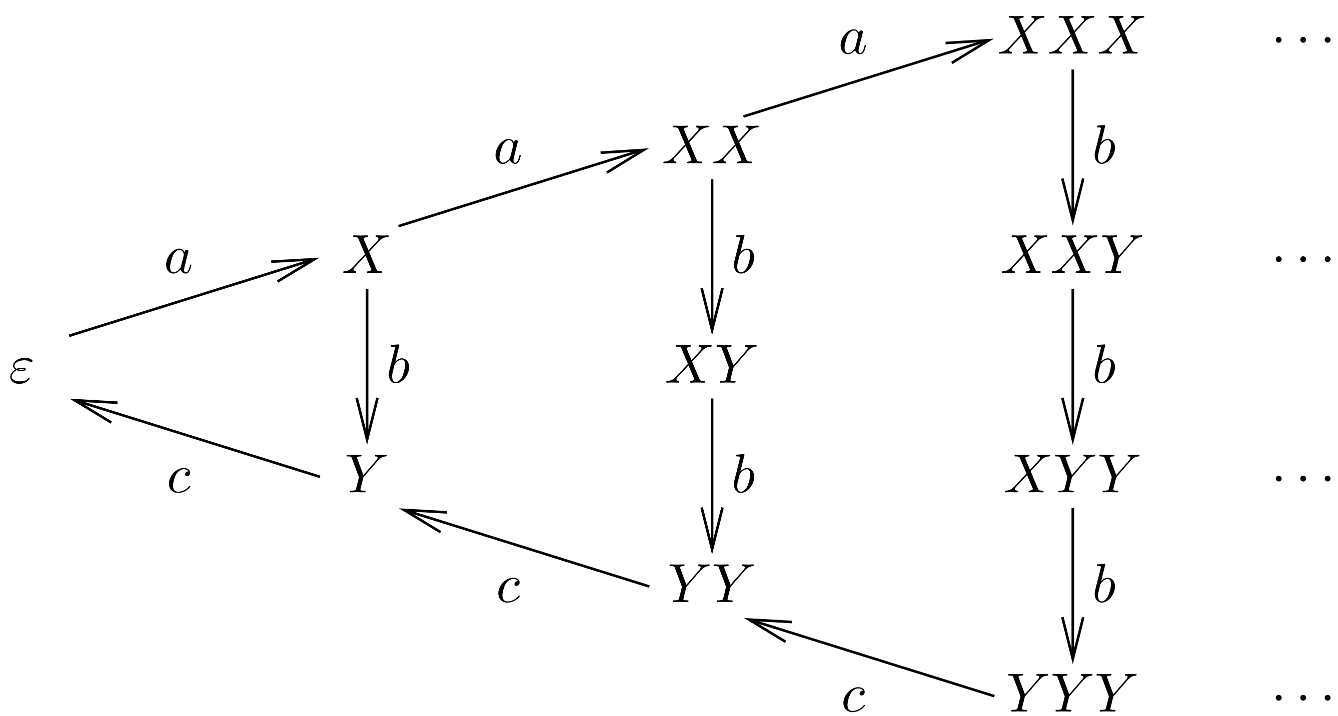
Synchronized Rational Graphs

Vertices: regular set V over alphabet Γ

Edges: synchronized rational relations, i.e. edge relation E_a is recognized by a *finite-state* automaton working on pairs $(X_1 \cdots X_m, Y_1 \cdots Y_n) \in \Gamma^* \times \Gamma^*$ with two *one-way* input tapes and *simultaneously* moving input heads.



Synchronized Rational Graphs: Example



Monotonic-Counter Extensions of Synchronized Rational Graphs

$(\Sigma \times D)$-labeled synchronized rational graph

$G := (V, (E_{(a,\bar{d})})_{(a,\bar{d}) \in \Sigma \times D})$

($D \subseteq \mathbb{N}^n$ finite, nonempty)

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Σ-labeled SRMC graph

$\widetilde{G} := (\widetilde{V}, (\widetilde{E}_a)_{a \in \Sigma})$ with $\widetilde{V} := V \times \mathbb{N}^n$

and $((\alpha, \bar{x}), (\beta, \bar{y})) \in \widetilde{E}_a$ iff

$(\alpha, \beta) \in E_{(a,\bar{d})}$ and

$\bar{y} = \bar{x} + \bar{d}$, for some $\bar{d} \in D$.

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$\bar{y} = \bar{x} + \bar{d}$, for some $\bar{d} \in D$.

Proposition: $\widetilde{G}$ is synchronized rational.

Proof sketch. Encode vertex $(\alpha, (x_1, \ldots, x_n))$ of $\widetilde{G}$ by means of word

$$\underbrace{\#_1 \cdots \#_1}_{x_1} \cdots \underbrace{\#_n \cdots \#_n}_{x_n} \; \alpha$$

Define automaton for $\widetilde{E}_a$ working on pairs $(\#_1^{x_1} \cdots \#_n^{x_n} \alpha, \#_1^{y_1} \cdots \#_n^{y_n} \beta)$:

1. Guess a vector $\bar{d} \in D$ and check whether $\bar{x} + \bar{d} = \bar{y}$.
2. Simulate the automaton for $E_{(a,\bar{d})}$ on (α, β).

Bounded delay sufficient since D and Γ are finite. $\square$

Part 3

Reachability Problem

Logical Decision Problems over Transition Graphs

- *Reachability*:

  Given a graph G and two vertices α and β in G, is β reachable from α?

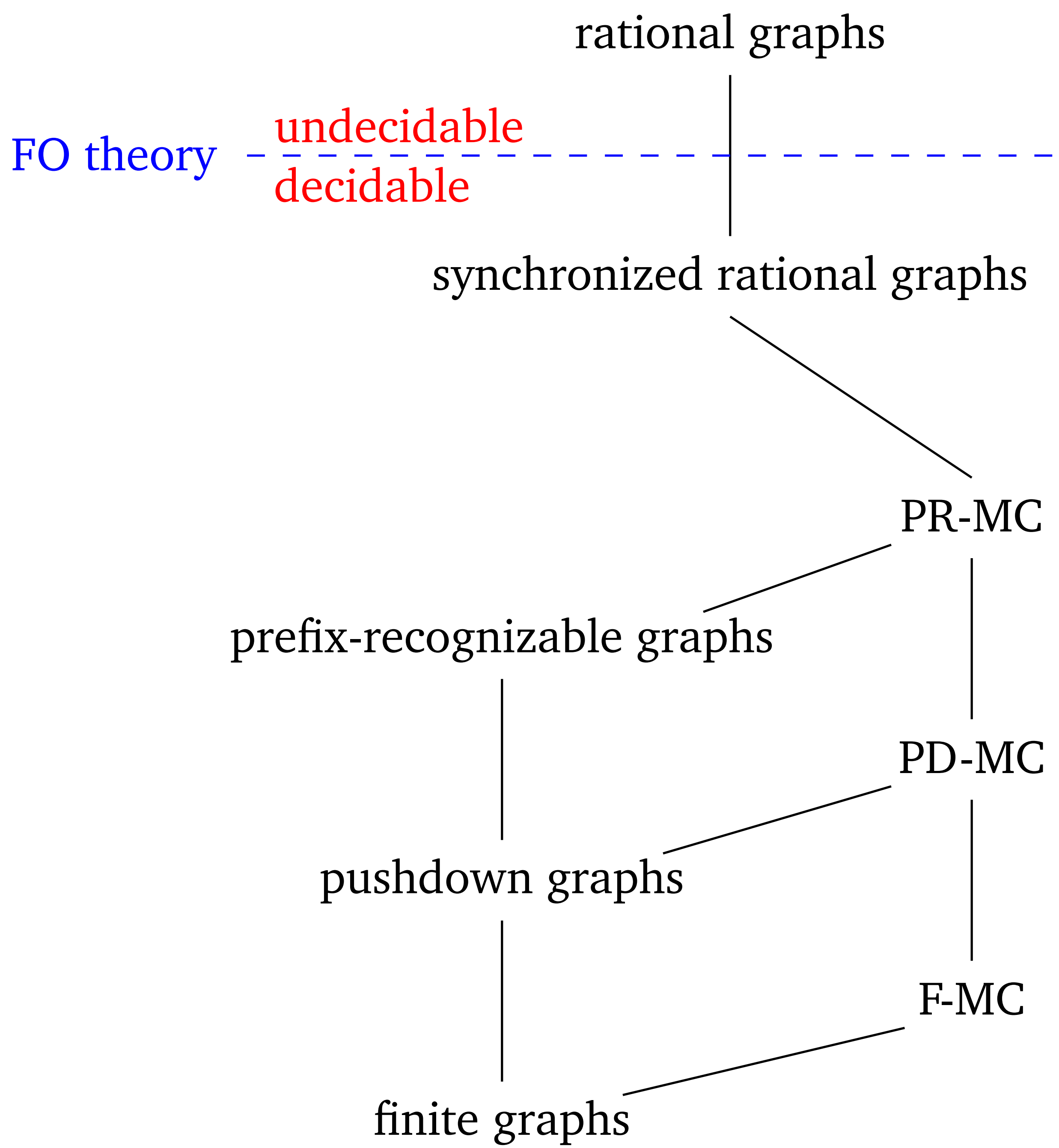
- *First-order (FO) theory*:

  Given a graph G and a first-order sentence φ, does φ hold in G?

- *Monadic second-order (MSO) theory*:

  Given a graph G and a monadic second-order sentence φ, does φ hold in G?

Logical Decision Problems over Transition Graphs



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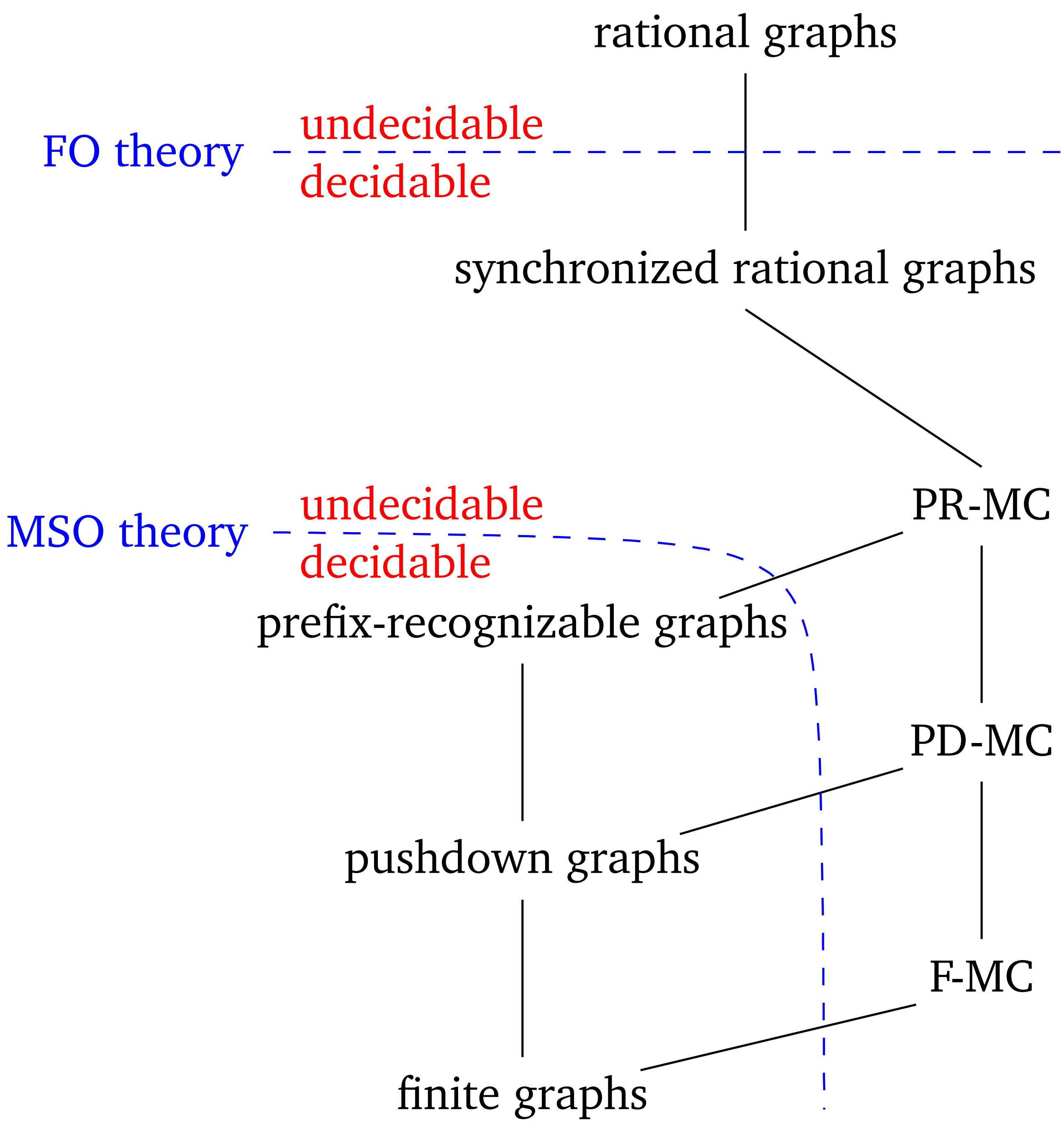

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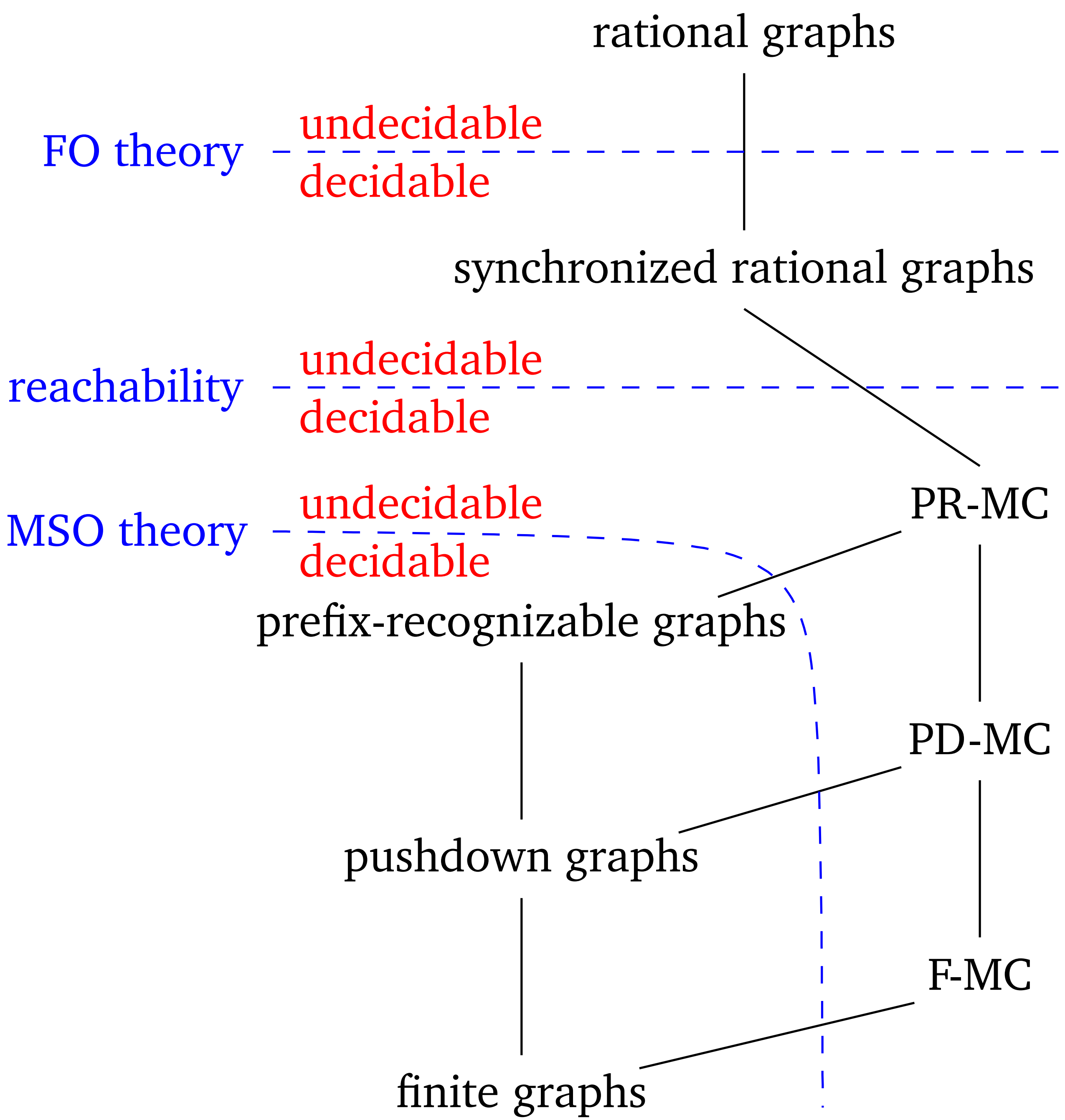


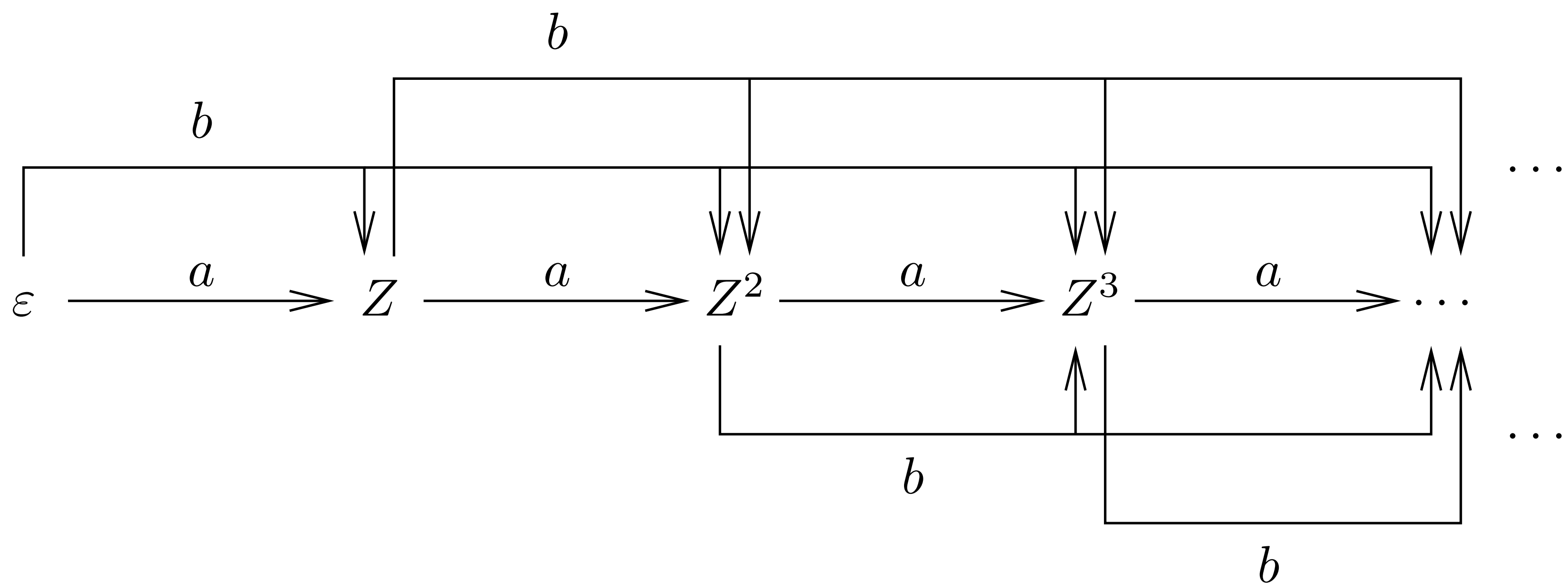
$\Sigma := \{a, b\}$ $\Gamma := \{Z\}$

rule $U \xrightarrow{a} V$ *with* $U, V \subseteq \Gamma^*$ *regular*

Prefix-rewriting system $\mathfrak{R} := \{\ \varepsilon \xrightarrow{a} Z\ ,\ \varepsilon \xrightarrow{b} Z^+\ \}$

Prefix-recognizable graph $G = (V, E_a, E_b)$ defined by $\mathfrak{R}$:

- $V = \Gamma^*$,
- $E_a = \{(Z^i, Z^{i+1}) \mid i \in \mathbb{N}\}$, and
- $E_b = \{(Z^i, Z^j) \mid i, j \in \mathbb{N}$ and $i < j\}$.



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Monotonic-Counter Extensions of Prefix-Recognizable Graphs

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Reachability problem for $\widetilde{G}$:

Given: *regular* sets $U, U' \subseteq V$ of vertices in G and
semi-linear sets $C, C' \subseteq \mathbb{N}^n$

Question: are there vertices $(\alpha, \bar{x}) \in U \times C$ and
$(\beta, \bar{y}) \in U' \times C'$ in $\widetilde{G}$ such that
$(\beta, \bar{y})$ is reachable from $(\alpha, \bar{x})$?

Proposition: The reachability problem for the monotonic-counter extension of any prefix-recognizable graph is decidable.

Proof sketch. Let $L \subseteq (\Sigma \times D)^*$ be the *traces* of G with U and U' as the set of initial and final vertices, respectively.

Show

$$U' \times C' \text{ is reachable from } U \times C \quad \text{iff} \quad (C + \Phi(L)) \cap C' \neq \emptyset$$

Right-hand side:

– L is context-free and effectively constructible (Caucal 2003).

– $\Phi(L)$ is effectively semi-linear.

– Effective closure of semi-linear sets under $+$ and $\cap$.

$\Longrightarrow$ (Non-)Emptiness is decidable

□

Summary

- Parikh automata correspond to adding monotonic counters with an additional semi-linearity test.
- No increase in language recognition power for linear-bounded automata, in contrast to finite and pushdown automata.
- Application to transition graphs: no increase for synchronized rational graphs, but for pushdown and prefix-recognizable graphs.
- For prefix-recognizable graphs, reachability remains decidable.

Further Prospects

- Using more general frameworks than monotonic counters, e.g. reversal-bounded counters.
- Allowing intermediate tests during computations.
- Comparing Parikh automata to reversal-bounded counter automata.
- Using more general arithmetical conditions than semi-linear sets.