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On the Parikh Images of Level-Two Pushdown Automata

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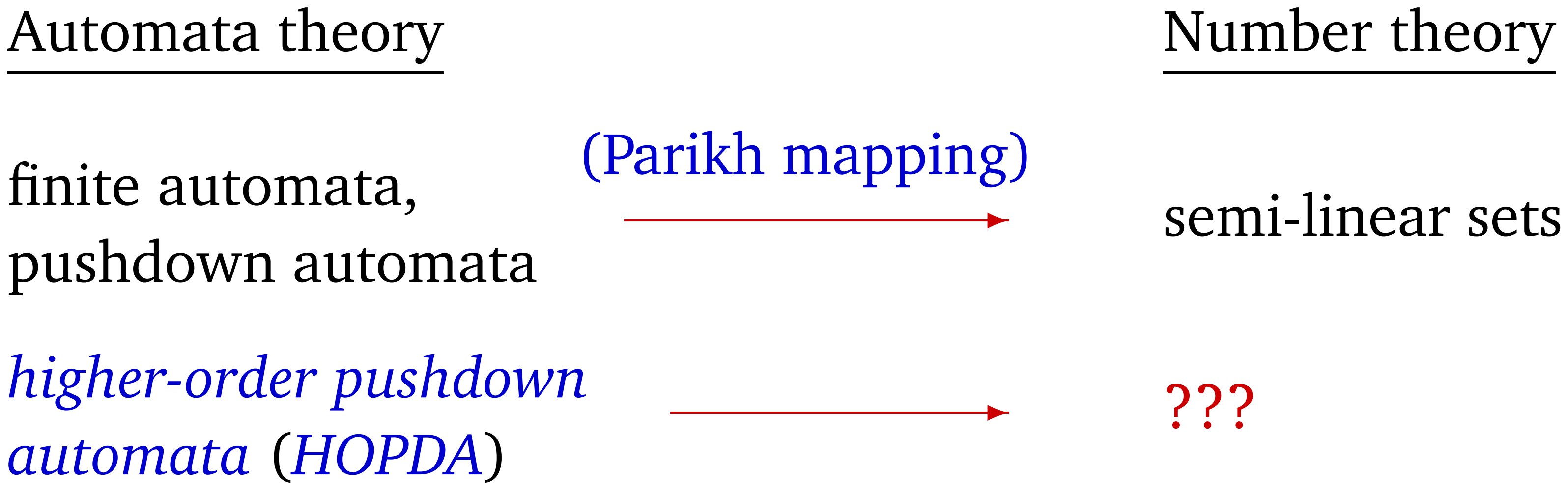

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finite automata,
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higher-order pushdown automata (*HOPDA*) →

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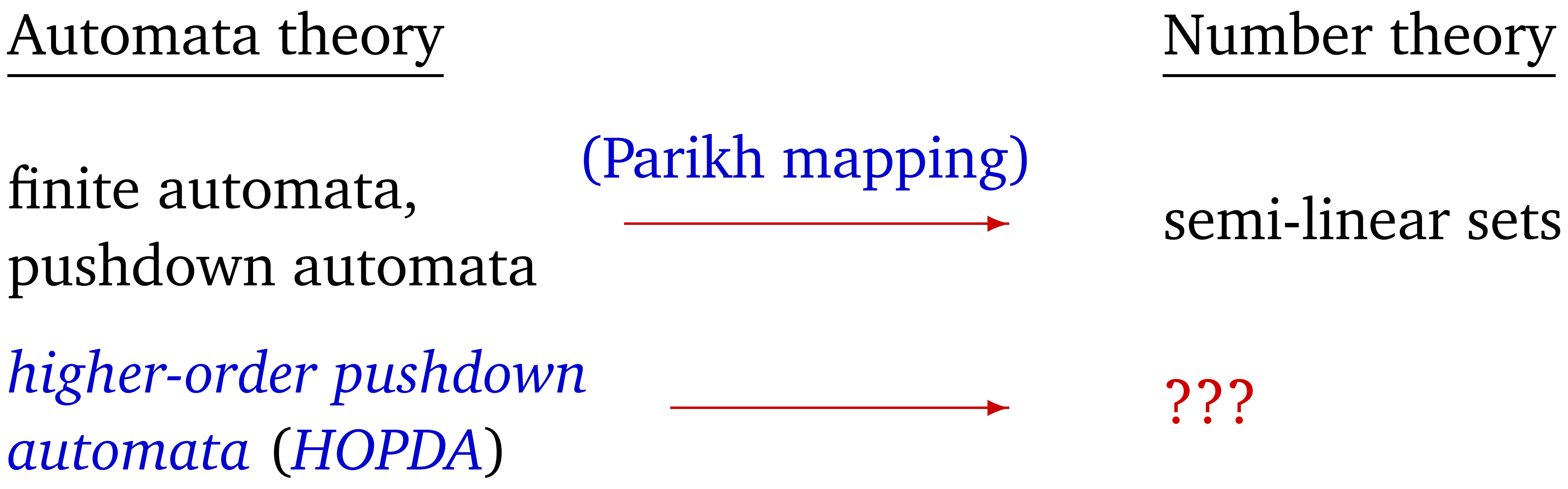

Automata theory

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HOPDA: finite-state automata with *a stack of stacks of . . . of stacks*

In this talk: *HOPDA of level 2* (*2-PDA*)

Two questions for a class characterizing the Parikh images of 2-PDA's:

- Can all sets from this class be generated (via the Parikh mapping)?
- Does the Parikh image of each 2-PDA belong to this class?

Outline

- Semi-linear sets and Parikh’s theorem
- Level 2 pushdown automata
- Semi-polynomial sets
- From semi-polynomial sets to 2-PDA’s
- From 2-PDA’s to semi-polynomial sets?
- Conclusions

Semi-Linear Sets

$A \subseteq \mathbb{N}^n$ *linear*: $A = \{\bar{x}_0 + k_1\bar{x}_1 + \ldots + k_m\bar{x}_m \mid k_1, \ldots, k_m \in \mathbb{N}\}$

for some $\bar{x}_0$, $\underbrace{\bar{x}_1, \ldots, \bar{x}_m}$ $\in \mathbb{N}^n$

constant vector periods

Semi-linear set: finite union of linear sets.

Example: $B := \{(x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_1 < x_2 < x_3\}$ is linear:

$$\{(0,1,2) + k_1(0,0,1) + k_2(0,1,1) + k_3(1,1,1) \mid k_1, k_2, k_3 \in \mathbb{N}\}\,.$$

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Properties of semi-linear sets:

- effective closure under Boolean operations [Ginsburg & Spanier]
- equivalence to Presburger-definable sets [Ginsburg & Spanier]

Parikh Mapping and Parikh's Theorem

$\Sigma = \{a_1, \ldots, a_n\}$

- *Parikh mapping* $\Phi\colon \Sigma^* \to \mathbb{N}^n$

$$\Phi(w) := (|w|_{a_1}, \ldots, |w|_{a_n}) \, .$$

- $\Phi(w)$: the *Parikh image* of w
- $\Phi(L) := \{\Phi(w) \mid w \in L\} \subseteq \mathbb{N}^n$: the *Parikh image* of L

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Theorem (Parikh (1961)): The Parikh image of any context-free language is effectively semi-linear.

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- Finite-state automata augmented with a nested pushdown stack, i.e., a stack of stacks of . . . stacks
- Level n HOPDA: n-fold nested stacks

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- Finite-state automata augmented with a nested pushdown stack, i.e., a stack of stacks of . . . stacks
- Level n HOPDA: n-fold nested stacks
- Background:
  - Maslov (1976): Formal definition; correspondence with generalized indexed languages
  - Damm and Goerdt (1982): automaton characterization of the OI hierarchy
  - Engelfriet (1983): correspondence to complexity classes
  - Carayol and Wöhrle (2003): correspondence to a hierarchy of infinite graphs introduced by Caucal

Level 2 PDA

Stack alphabet Γ with initial symbol $\perp$:

- Level 1 stack (1-stack): $[Z_m \cdots Z_1]$; Z_m is the topmost symbol.
- Level 2 stack (2-stack): $[s_r, \ldots, s_1]$, where $s_1, \ldots, s_r$ are 1-stacks, and s_r is the topmost 1-stack.
- Empty level 2 stack $[[\varepsilon]]$; initial level 2 stack $[[\perp]]$.

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- Empty level 2 stack $[[\varepsilon]]$; initial level 2 stack $[[\perp]]$.
- Instructions on 1-stacks: push and pop
- Instructions on level 2 stacks:
  - push and pop on the topmost 1-stack
  - *copy the topmost 1-stack*
  - *remove the topmost 1-stack*
- Access: *only to the topmost symbol of the topmost 1-stack !*

Example: A Non-Semi-Linear Language

$L_{\text{quad}} := \{a^k b^{k^2} \mid k \in \mathbb{N}\} \subseteq \{a, b\}^*$

Take $\Gamma := \{\perp, Z, Z_2\}$ and process input $a^k b^{k^2}$ as follows:

$$[[Z^{2k}\perp]]$$

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$$\begin{array}{rcr} [[Z & Z^{2(k-1)} & \perp], \\ [Z_2 & Z^{2k} & \perp]] \end{array}$$

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$\Phi(L_{\text{quad}}) = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1^2 = x_2\}$

$\Longrightarrow$ not semi-linear (proof by growth rate arguments)

Semi-Polynomial Sets

- $A \subseteq \mathbb{N}^n$ linear: $\bar{x} \in A$ iff $k_1, \ldots, k_m \in \mathbb{N}$ exist such that

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- Replace the vectors with their components:

$$\bar{x} = (x_{01}, \ldots, x_{0n}) + (k_1 x_{11}, \ldots, k_1 x_{1n}) + \ldots + (k_m x_{m1}, \ldots, k_m x_{mn})$$

and *replace linear terms $k_i x_{ij}$ with polynomial ones*.

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- $A \subseteq \mathbb{N}^n$, $n \geq 1$, is called *polynomial of degree* $d \geq 1$ if there exist $\bar{x}_0$ (the *constant*) and $(\bar{x}_{i,j})_{1 \leq i \leq m, 1 \leq j \leq d}$ (the *periods*) with

$$\begin{aligned} A = \{\bar{x}_0 + k_1\bar{x}_{1,1} + k_1^2\bar{x}_{1,2} + \cdots + k_1^{d-1}\bar{x}_{1,d-1} + k_1^d\bar{x}_{1,d} \\ + \cdots + k_m\bar{x}_{m,1} + k_m^2\bar{x}_{m,2} + \cdots + k_m^{d-1}\bar{x}_{m,d-1} + k_m^d\bar{x}_{m,d} \\ \mid k_1, \ldots, k_m \in \mathbb{N}\}\,. \end{aligned}$$

  Semi-polynomial set of degree d: finite union of polynomial sets of degree d

From Semi-Polynomial Sets to 2-PDA’s

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A technical lemma: decomposition of polynomials

Lemma: Let $k \geq 0$ and $e > 0$. Then,

$$k^e = \sum_{i=0}^{k-1} \left(c_{e-1} i^{e-1} + c_{e-2} i^{e-2} + \cdots + c_2 i^2 + c_1 i + 1 \right) , \tag{1}$$

where $c_j := \binom{e}{j}$ for $j = 1, \ldots, e-1$.

Proof. By induction on k. □

The Core Construction

Lemma: Let $d \geq 1$ and $\Gamma := \{\perp, Z_1, \ldots, Z_d\}$. Then, for $1 \leq e \leq d$, $k \in \mathbb{N}$, and $w \in \Sigma^*$, we can construct a 2-PDA $\mathfrak{A}$ with states p and q which proceeds from configuration $(p, [[Z_d Z^{2k} \perp], s_r, \ldots, s_1])$ to configuration $(q, [[Z_d Z^{2k} \perp], s_r, \ldots, s_1])$ after reading w^{k^e}.

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Proof. By induction on d.

For simplicity, let $Z := Z_1$.

$d = 1$: Only subcase $e = 1 \Rightarrow k^e = k$.

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$d = 2$: For $e = 1$, as before. For $e = 2$, as with L_{quad}.

$d \geq 3$: For $e = 1, 2$, as before.

For $e = 3, \ldots, d$, apply Decomposition Lemma:

$$k^e = \sum_{i=0}^{k-1} \left(c_{e-1} i^{e-1} + c_{e-2} i^{e-2} + \cdots + c_2 i^2 + c_1 i + 1 \right) ,$$

where $c_j := \binom{e}{j}$, for $j = 1, \ldots, e-1$.

$$[[Z_d \qquad\qquad Z^{2k} \qquad\qquad \bot], \ldots].$$

$$\begin{array}{lll} [[Z_{e-1} & Z^{2(k-1)} & \perp], \\ [Z_d & Z^{2k} & \perp], \ldots]\,. \end{array}$$

$$\begin{array}{r}
[[Z_{e-1}\bot], \\
[Z_{e-1}ZZ\bot], \\
[Z_{e-1}ZZZZ\bot], \\
\vdots \qquad\qquad\qquad \\
[Z_{e-1} \qquad Z^{2(k-1)} \qquad \bot], \\
[Z_d \qquad\qquad Z^{2k} \qquad\qquad \bot], \ldots]\,.
\end{array}$$

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\vdots \\
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[Z_d \qquad\qquad\qquad Z^{2k} \qquad\qquad\quad \bot], \ldots].
\end{array}$$

For each $i = 0, \ldots, k-1$, starting from 1-stack $[Z_{e-1}Z^{2i}\bot]$, process

$$(c_{e-1}i^{e-1} + c_{e-2}i^{e-2} + \cdots + c_2 i^2 + c_1 i + 1)\,.$$

successive words w, using the induction hypothesis.

The procedure ends if Z_d appears, having processed

$$\sum_{i=0}^{k-1} \left(c_{e-1}i^{e-1} + c_{e-2}i^{e-2} + \cdots + c_2 i^2 + c_1 i + 1\right) \ ,$$

i.e. (k^e)-many successive words w. □

Main Theorem

Theorem: Every semi-polynomial set is the Parikh image of a 2-PDA-recognizable language.

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Proof. W.l.o.g., consider only polynomial sets (closure under union).

Let $A \subseteq \mathbb{N}^n$ be a polynomial set of degree d, given by $\bar{x}_0$ and $\bar{x}_{i,j}$ $(1 \leq i \leq m,\ 1 \leq j \leq d)$.

Define $\Sigma := \{a_1, \ldots, a_n\}$ and assign to $\bar{x}_0$ and $\bar{x}_{i,j}$ words w_0 and $w_{i,j}$ in $a_1^* \cdots a_n^*$.

Construct 2-PDA $\mathfrak{A}$ with

$$\begin{aligned} L(\mathfrak{A}) = \ & \{w_0 \\ & \quad w_{1,1}^{k_1} w_{1,2}^{k_1^2} \cdots w_{1,d-1}^{k_1^{d-1}} w_{1,d}^{k_1^d} \\ & \quad \ldots \\ & \quad w_{m,1}^{k_m} w_{m,2}^{k_m^2} \cdots w_{m,d-1}^{k_m^{d-1}} w_{m,d}^{k_m^d} \mid k_1, \ldots, k_m \in \mathbb{N}\} \,. \end{aligned}$$

Main Theorem (continued)

- Read w_0 (without using the stack).
- For $i = 1, \ldots, m$:
  - Guess k_i by pushing $(2k_i)$-many Z's followed by a Z_d, resulting in $[[Z_d Z^{2k_i} \bot]]$.
  - For $j = 1, \ldots, d$: By previous lemma, process (k_i^j)-many successive words $w_{i,j}$.
  - After having processed
  $$w_{i,1}^{k_i} w_{i,2}^{k_i^2} \cdots w_{i,d-1}^{k_i^{d-1}} w_{i,d}^{k_i^d},$$
  remove k_i from the stack and proceed with next i.
- The procedure ends after we have processed

$$w_0 \quad w_{1,1}^{k_1} w_{1,2}^{k_1^2} \cdots w_{1,d-1}^{k_1^{d-1}} w_{1,d}^{k_1^d} \quad \cdots \quad w_{m,1}^{k_m} w_{m,2}^{k_m^2} \cdots w_{m,d-1}^{k_m^{d-1}} w_{m,d}^{k_m^d}.$$

□

From 2-PDA to Semi-Polynomial Sets? (1)

$L_{\mathrm{exp}} := \{a^x b^{2^x} \mid x \in \mathbb{N}\}$ is 2-PDA-recognizable (Carayol and Wöhrle):

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Implement a *binary counting* using top symbols of 1-stacks as bits.

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Implement a *binary counting* using top symbols of 1-stacks as bits.

However, $\Phi(L_{\text{exp}}) = \{(x, 2^x) \mid x \in \mathbb{N}\}$ is not semi-polynomial ! (proof by growth rate arguments).

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$$\left.\begin{array}{c} [[Z^x \bot], \\ \vdots \\ [Z^x \bot], \\ [Z^x \bot]] \end{array}\right\} \quad y\text{-times}$$

Number of Z's: xy

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$L_{\mathrm{prod}} := \{a^x b^y c^{xy} \mid x, y \in \mathbb{N}\}$ is 2-PDA-recognizable:

Take $\Gamma := \{\perp, Z\}$ and process input $a^x b^y c^{xy}$ as follows:

$$\left.\begin{array}{c} [[Z^x \perp], \\ \vdots \\ [Z^x \perp], \\ [Z^x \perp]] \end{array}\right\} \quad y\text{-times}$$

Number of Z's: xy

However, $\Phi(L_{\mathrm{prod}}) = \{(x, y, xy) \mid x, y \in \mathbb{N}\}$ is not semi-polynomial !
(the proof involves a number-theoretical analysis)

Summary and Further Prospects

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- Semi-Polynomial sets *cannot* capture the Parikh images of 2-PDA's.
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  - product relations
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Future work:

- Extending semi-polynomial sets to capture the Parikh images of 2-PDA's (n-PDA's).
- Restricting 2-PDA's such that only semi-polynomial sets are generated.