Infinite Games and Uniformization

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Abstract. The problem of solvability of infinite games is closely connected with the classical question of uniformization of relations by functions of a given class. We work out this connection and discuss recent results on infinite games that are motivated by the uniformization problem.

The fundamental problem in the effective theory of infinite games was posed by Church in 1957 ("Church's Problem"; see [2, 3]). It refers to two-player games in the sense of Gale and Stewart [6] in which the two players 1 and 2 build up two sequences $\alpha = a_0a_1\ldots$, respectively $\beta = b_0b_1\ldots$, where a_i, b_i belong to a finite alphabet Σ. Player 1 picks a_0, then Player 2 picks b_0, then Player 1 picks a_1, and so forth in alternation. A play is an ω-word over $\Sigma \times \Sigma$ of the form $\binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\ldots$; we also write $\alpha^\frown\beta$. A game is specified by a relation $R \subseteq \Sigma^\omega \times \Sigma^\omega$, or equivalently by the ω-language $L_R = \{\alpha^\frown\beta \mid (\alpha, \beta) \in R\}$. Player 2 wins the play $\alpha^\frown\beta$ if $(\alpha, \beta) \in R$.

Church's Problem asks for a given relation R ("specified in some suitable logistic system" [2]) whether Player 2 has a winning strategy in the game defined by R, i.e., whether there is a corresponding function f mapping finite play prefixes $\binom{a_0}{b_0}\binom{a_1}{b_1}\ldots\binom{a_n}{*}$ to the set Σ, providing the information which letter to pick next (and – if there is a winning strategy f – to provide a definition of f). A strategy f induces a function $f' : \Sigma^\omega \to \Sigma^\omega$; for a winning strategy f we then have $(\alpha, f'(\alpha)) \in R$ for all $\alpha \in \Sigma^\omega$.

A prominent class of games is given by the regular (or equivalently: the MSO-definable) relations R, which we identify here with the associated regular ω-languages L_R. In this case a complete solution is known by the "Büchi-Landweber Theorem" ([1]; see e.g. [7]): Given a regular game (specified, e.g., by a Büchi automaton or an MSO-formula),

- either of the players has a winning strategy (the game is "determined"),
- one can decide who is the winner,
- and one can construct a finite-state strategy for the winner (i.e., a strategy realizable by a Mealy automaton).

The extraction of a function f from a relation R such that for each argument x we have $(x, f(x)) \in R$ is called "uniformization". More precisely, a class $\mathcal{R}$ of binary relations $R \subseteq D \times D'$ is *uniformizable* by functions in a class $\mathcal{F}$ if for each $R \in \mathcal{R}$ there is a function $f \in \mathcal{F}$ such that the graph of f is contained in R, and the domains of R and f coincide (see Fig. 1 for an illustration).

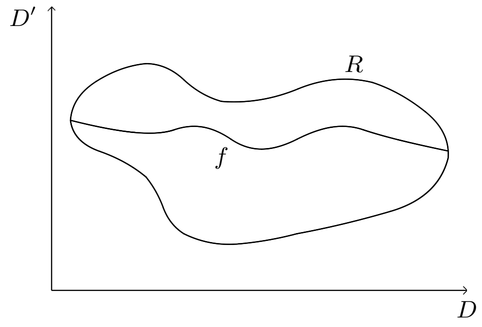

Fig. 1. Uniformization

Two well-known examples from recursion theory and from automata theory are concerned with the recursively enumerable, respectively the rational relations; here we have the "ideal" case that the graphs of the required functions are precisely of the type of the given relations.

For the first example, let us recall that a partial function from $\mathbb{N}$ to $\mathbb{N}$ is recursive iff its graph is recursively enumerable. The Uniformization Theorem of recursion theory says that a binary recursively enumerable relation R is uniformizable by a function whose graph is again recursively enumerable, i.e. by a (partial) recursive function f. (A computation of $f(x)$ works as follows: Enumerate R until a pair (y, z) is reached with $x = y$, and in this case produce z as output.)

For the second example, recall that a binary rational word relation is defined (for instance) in terms of a finite nondeterministic two-tape automaton that scans a given word pair (u, v) asynchronously, i.e. with two reading heads that move independently from left to right over u, respectively v. Rational relations are uniformizable by rational functions, defined as the functions whose graph is a rational relation (see e.g. [4, 10]).

In the task of uniformization as it appears in Church's Problem, there are two special features: First, one looks for functions that are computed "online" (step by step in terms of the argument). Secondly, determinacy results constitute a very special situation where non-existence of a function f with $(x, f(x)) \in R$ for all x implies the existence of a function g such that $(g(y), y) \notin R$ for all y.

The first requirement can be weakened in the sense that the functions f used for uniformization could use more information than just $a_0 \dots a_i$ for producing the letter b_i. Different types of "look-ahead" can be studied, the extreme case being that the whole sequence $\alpha = a_0 a_1 \dots$ is given when $\beta = b_0 b_1 \dots$ is to be formed. Finite look-ahead corresponds to the condition that the i-th letter b_i of β depends only on a finite prefix of $\alpha = a_0 a_1 \dots$; one calls such functions "continuous" (in the Cantor topology over Σ^ω). We discuss recent results of [8] for regular games: If the uniformization of a regular relation R is possible with a continuous function, then a function of bounded look-ahead k suffices, where, for all i, the letter b_i depends only on the prefix $a_0 \dots a_{i+k}$.

The Büchi-Landweber Theorem can be phrased as saying that MSO-definable strategies suffice for solving MSO-definable games, or – in other words – that MSO-definable relations can be uniformized by MSO-definable functions. (Here we use a notion of definability of functions in terms of arguments given as finite words.)

This motivates a study of Church's Problem for other classes of relations. We start with relations that are first-order definable rather than MSO-definable. There are two natural versions of first-order logic, denoted FO(+1) and FO(<), where the items in brackets indicate the available arithmetical signature. We recall results of [9] where it is shown that a determinacy theorem holds for the FO(+1)-, respectively the FO(<)-definable relations, and that appropriate winning strategies exist which are again FO(+1)-, respectively FO(<)-definable. Continuing this track, we exhibit cases where this transfer fails (Presburger arithmetic is an example), and then address corresponding results of Fridman [5] for non-regular games that are defined by various types of ω-pushdown automata.

These investigations are small steps towards a more comprehensive understanding of uniformization problems in the context of infinite games. So far, general conditions are missing that "explain" the known scattered results.

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