Definability Questions for MSO

Christof Löding

RWTH Aachen

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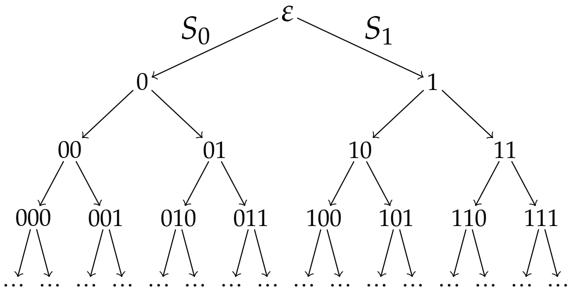
Decidable MSO Theories

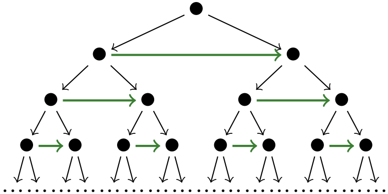
- $\mathcal{N} = (\mathbb{N}, S)$: natural numbers with successor (Büchi'62)

$$0 \xrightarrow{S} 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

- $\mathcal{T}_2 = (\{0,1\}^*, S_0, S_1)$: infinite binary tree (Rabin'69)



Adding New Relations



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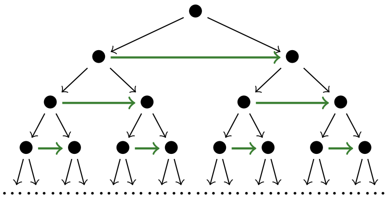

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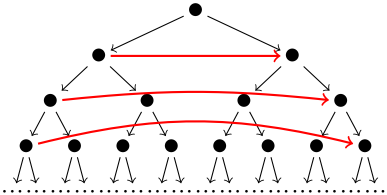

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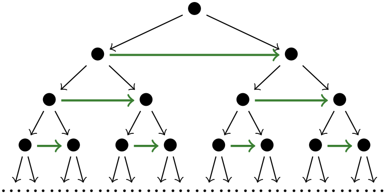

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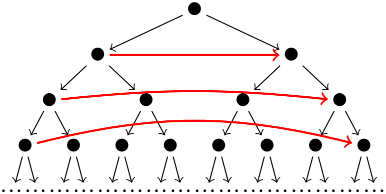
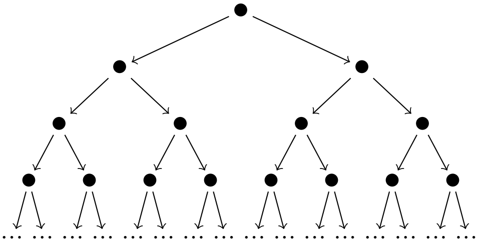


The red relation is not MSO definable in $\mathcal{T}_2$.

MSO Interpretations

Define a new structure in a given one by MSO formulas.



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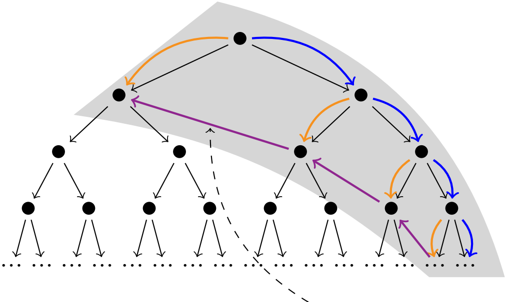

$$\phi(x, y) = \exists x', y' : S_0(x', x) \wedge S_0(y', y) \wedge S_1(y', x')$$

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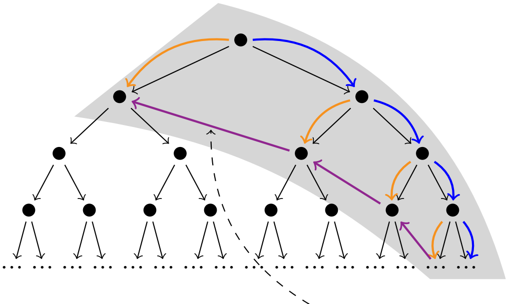

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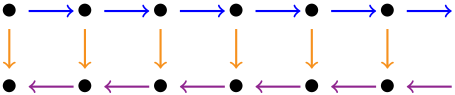

Notation

For structures $\mathcal{A}$, $\mathcal{B}$ define

$$\mathcal{A} \leq_{\mathrm{MSO}} \mathcal{B} \text{ iff } \mathcal{A} \text{ is MSO-interpretable in } \mathcal{B}$$

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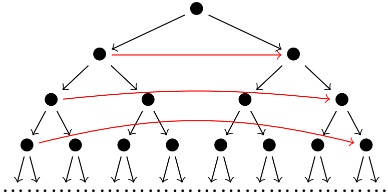
Observation for $\mathcal{A} \leq_{\mathrm{MSO}} \mathcal{B}$:

- If $\mathcal{B}$ has decidable MSO theory, then $\mathcal{A}$ has decidable MSO theory.
- If $\mathcal{A}$ has undecidable MSO theory, then $\mathcal{B}$ has undecidable MSO theory.

Example

Interpreting  in $\mathcal{T}_2$

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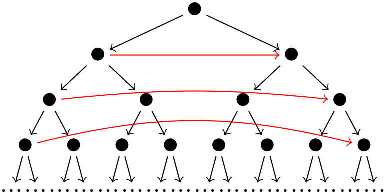
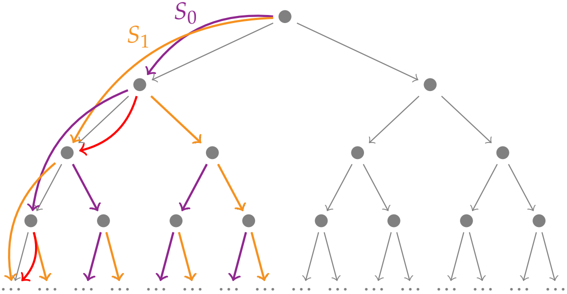
Interpreting  in $\mathcal{T}_2$



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- What kind of structures are MSO interpretable in it?

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Observations:

- If R is MSO definable in $\mathcal{A}$, then $(\mathcal{A}, R) \leq_{\text{MSO}} \mathcal{A}$.
- There is a relation R on $\mathcal{T}_2$ such that R is not MSO definable in $\mathcal{T}_2$ but $(\mathcal{T}_2, R) \leq_{\text{MSO}} \mathcal{T}_2$.

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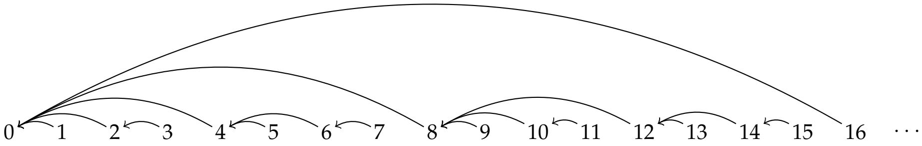
In $\mathcal{N}$ reorganizing the structure does not help for defining new relations, even when embedding $\mathcal{N}$ into $\mathcal{T}_2$:

Proposition (Colcombet/L.07). If $(\mathcal{N}, R) \leq_{\mathrm{MSO}} \mathcal{T}_2$, then R is already MSO definable in $\mathcal{N}$.

Example

The *flip* function (flip the least significant 1-bit in binary representation to 0):



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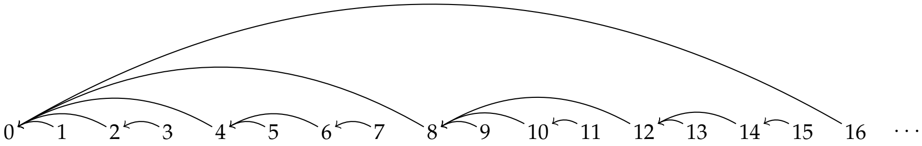

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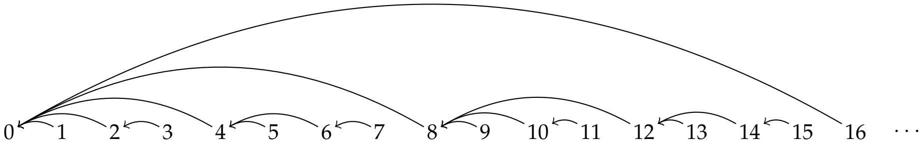

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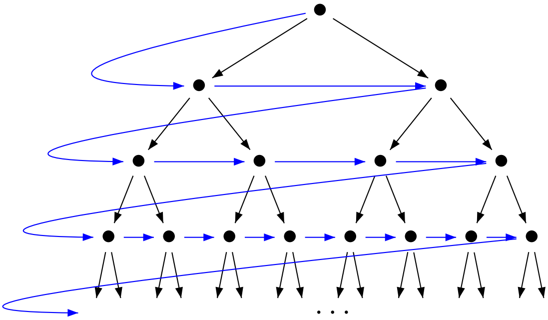
What about the other direction: Can we obtain $\mathcal{T}_2$ by interpretation in $(\mathcal{N}, \mathit{flip})$ or any other extension of $\mathcal{N}$ with decidable MSO theory?

Well-Orderings

Total order without infinite decreasing chains (each set has a minimal element)

Example: length lexicographic order $<_{\text{llex}}$



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Question: Is there an MSO-definable well-ordering over $\mathcal{T}_2$?

Or is $(\mathcal{T}_2, <) \leq_{\mathrm{MSO}} \mathcal{T}_2$ for some well-ordering $<$?

Or is there some well-ordering $<$ such that $(\mathcal{T}_2, <)$ has a decidable MSO theory?

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Theorem (Carayol/L. 07). Every extension $(\mathcal{T}_2, <)$ of the infinite binary tree with a well-ordering $<$ has an undecidable MSO theory.

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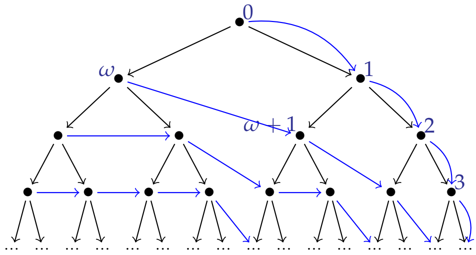

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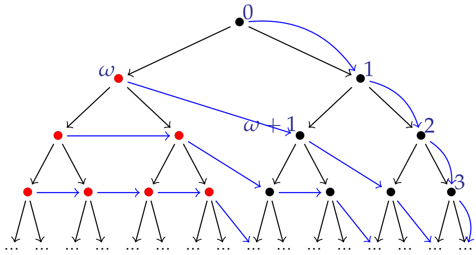

Consequences

Corollary. If a structure has a decidable MSO theory and admits an MSO-definable well-ordering, then $\mathcal{T}_2$ is not MSO interpretable in this structure. In particular, $\mathcal{T}_2$ is not MSO interpretable in any decidable extension of $\mathcal{N}$.

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$$\mathcal{T}_2 \not\leq_{\mathrm{MSO}} (\mathcal{N}, \mathit{flip})$$

Corollary. There is no extension of $\mathcal{N}$ by unary predicates such that $\mathcal{T}_2$ is MSO interpretable in this extension (even for extensions with undecidable MSO theory).

Uniformization

Uniformization asks for turning relations into functions.

Uniformization for MSO:

- Start from an MSO formula $\psi(\underbrace{X_1,\ldots,X_n}_{\overline{X}},\underbrace{Y_1,\ldots,Y_m}_{\overline{Y}})$ defining a relation between subsets.
- Construct a formula $\varphi(\overline{X},\overline{Y})$ such that
  - $\varphi(\overline{X},\overline{Y}) \rightarrow \psi(\overline{X},\overline{Y})$ and
  - $\forall \overline{X} : (\exists \overline{Y} : \psi(\overline{X},\overline{Y})) \rightarrow (\exists^1 \overline{Y} : \varphi(\overline{X},\overline{Y}))$

Uniformization for $\mathcal{N}$

Theorem (Siefkes'75) Uniformization for MSO is possible over $\mathcal{N}$.

Proof:

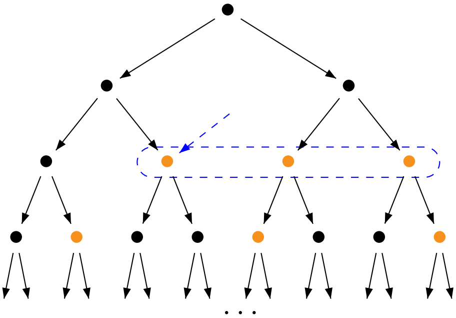
- Take a Büchi automaton equivalent to $\psi(\overline{X}, \overline{Y})$.
- For a fixed $\overline{X}$ pick the $\overline{Y}$ that is accepted by a "minimal" accepting run (visiting final states as early as possible).

What about uniformization for $\mathcal{T}_2$?

Choice functions

- General: Takes a (nonempty) set as argument and returns a unique element from this set.
- On the binary tree: Returns a unique node for each nonempty set of nodes.

  Example: Pick the minimal node according to $<_{\text{llex}}$.



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  MSO-definable: formula $\varphi(X, y)$
  For all (nonemtpy) sets U there is exactly one node $u \in U$ such that
  $$\mathcal{T}_2 \models \varphi[U, u]$$
- Choice is uniformization of the formula $y \in X$

Question: Is there an MSO-definable choice function on the binary tree?

Undefinability of choice

Theorem (Gurevich/Shelah'83). There is no MSO-definable choice function.

Proof: Complex, using tools from set theory

Carayol/L.'07: Simple proof using basic techniques from automata theory

Intuition: trying to define choice

- Try to find a formula $\varphi(X, y)$ defining a choice function.
- Convention: We identify a set of nodes with the tree $\mathcal{T}_2$ on which these nodes are colored.
- Formula $\varphi(X, y)$ can be transformed into a nondeterministic top-down tree automaton (Rabin'69)
- Can be seen as an automaton reading a tree with black and orange nodes
- In an accepting run the automaton has to mark one of the orange nodes (by moving to a special marking state)

First try

Pick the leftmost node.

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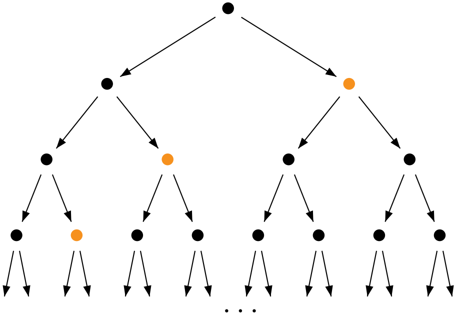

Second try

- If the leftmost branch contains ●, take the first.
- Ohterwise move to the first point on this branch where the right subtree contains ●.
- If the rightmost branch starting from this point contains ●, take the first.
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- ...

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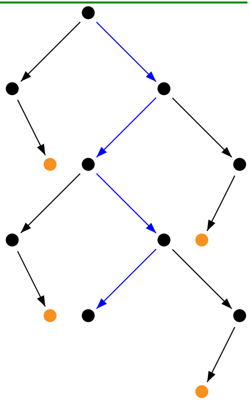
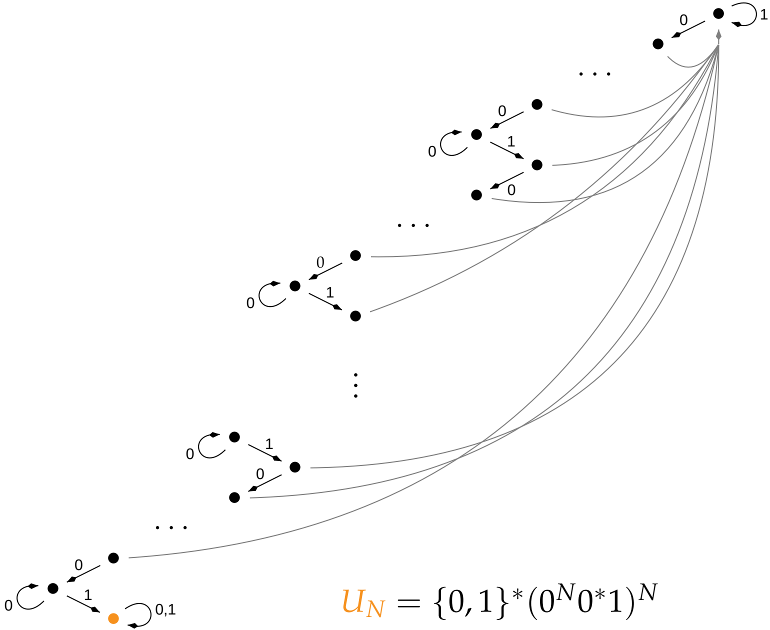


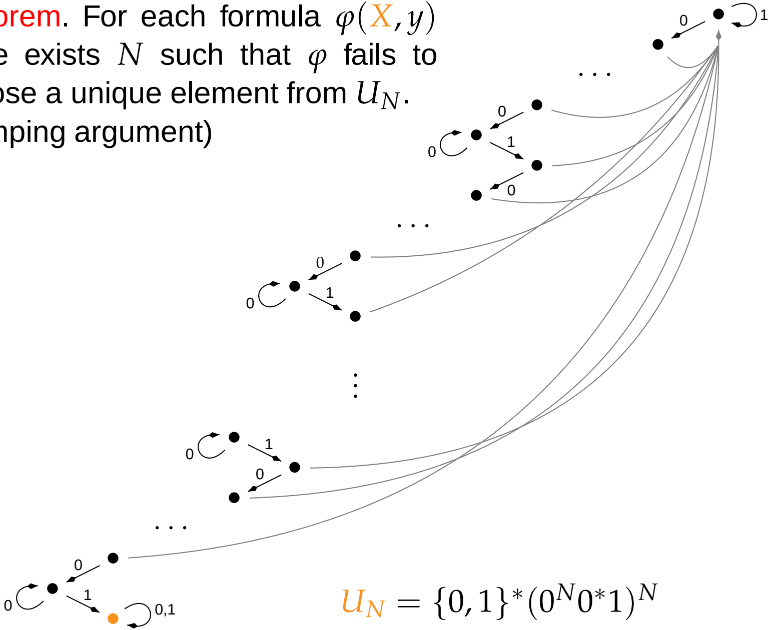
The counterexamples



$$U_N = \{0,1\}^*(0^N 0^* 1)^N$$

The counterexamples

Theorem. For each formula $\varphi(X, y)$ there exists N such that φ fails to choose a unique element from U_N. (pumping argument)



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Adding Unary Predicates

Corollary. There is no MSO-definable choice function in any extension of $\mathcal{T}_2$ by unary predicates.

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Proof:

- Assume $\varphi(X, y)$ defines a choice function in $(\mathcal{T}_2, P_1, \ldots, P_n)$.
- $\exists P_1, \ldots, P_n$: $\underbrace{\text{“}\varphi(X, y)\text{defines a choice function”}}_{\text{MSO definable}}$

  is satisfiable in $\mathcal{T}_2$.

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- $\exists P_1, \ldots, P_n$: "$\underbrace{\varphi(X,y) \text{defines a choice function}}_{\text{MSO definable}}$" is satisfiable in $\mathcal{T}_2$.
- Then there are regular $P_1, \ldots, P_n$ such that $\varphi(X,y)$ defines a choice function in $(\mathcal{T}_2, P_1, \ldots, P_n)$.
- Regular predicates are MSO definable and can be eliminated.

An Application: Unambiguous Automata

- Unambiguous automata are nondeterministic automata that have at most one accepting run for each input.
- Unambiguous automata exist for regular languages of finite words, finite trees, and infinite words.
- Unambiguous automata can be exponentially more succinct than deterministic ones.
- Equivalence is decidable in polynomial time for unambiguous. automata on finite words (Stearns/Hunt'85) and finite trees (Seidl'90)

Question: Do unambiguous automata for regular languages of infinite trees always exist?

Unambiguous Rabin Tree Automata

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Theorem (Niwiński/Walukiewicz). There is no unambiguous Rabin tree automaton for L_{amb}.

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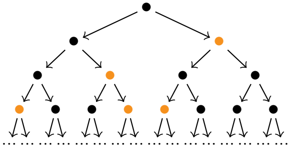
Proof:

- Give a tree over $\{a, b\}$, consider the set of b-nodes



- An accepting run has to verify that there is at least one b in the tree.
- If the automaton is unambiguous, then it picks for each set of b a unique one $\rightsquigarrow$ choice function.

Conclusion

- Every extension of $\mathcal{T}_2$ with a well-ordering leads to an undecidable MSO theory.
  - $\mathcal{T}_2$ is not MSO interpretable in any extension of $\mathcal{N}$ with decidable MSO theory.
- Uniformization is possible for $\mathcal{N}$.
- Uniformization fails for $\mathcal{T}_2$: there is no MSO-definable choice function.
  - Proof using simple automata theoretic techniques
  - Application: inherent ambiguity of Rabin tree automata

Some open questions:

- Cases for which uniformization is possible over $\mathcal{T}_2$.
  Example: formulas of type $\varphi(x, Y)$
- Deciding if a formula admits uniformization