Logical Refinements of Church's Problem *

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Abstract. Church's Problem (1962) asks for the construction of a procedure which, given a logical specification φ on sequence pairs, realizes for any input sequence X an output sequence Y such that (X, Y) satisfies φ. Büchi and Landweber (1969) gave a solution for MSO specifications in terms of finite-state automata. We address the problem in a more general logical setting where not only the specification but also the solution is presented in a logical system. Extending the result of Büchi and Landweber, we present several logics $\mathcal{L}$ such that Church's Problem with respect to $\mathcal{L}$ has also a solution in $\mathcal{L}$, and we discuss some perspectives of this approach.

1 Introduction

An influential paper in automata theory is the address of A. Church to the Congress of Mathematicians in Stockholm (1962) [3]. Church discusses systems of restricted arithmetic, used for conditions on pairs (X, Y) of infinite sequences over two finite alphabets. As "solutions" of such a condition φ he considers "restricted recursions" (of which the most natural example is provided by finite automata with output), which realize letter-by-letter transformations of input sequences X to output sequences Y, such that φ is satisfied by (X, Y). By "Church's Problem" one understands today the question whether a condition in MSO (the monadic second-order theory of the structure $(\mathbb{N}, <)$) can be realized in this sense by a finite automaton with output, and in this case to synthesize it. Büchi and Landweber solved this problem, and in recent years many authors took up the question in various applications of the algorithmic theory of infinite games (see e.g. [6]).

In the original problem, a precise relation between a specification and its solution was not addressed, maybe due to the fact that specifications and solutions were considered to be in different domains. However, by a well-known correspondence between finite automata and monadic second-order logic (established by Büchi, Elgot, and Trakhtenbrot), the Büchi-Landweber solution for MSO-specifications can again be presented as MSO-formulas. In this paper we

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analyze this view and generalize the result to a number of sublogics of MSO. We exhibit natural logics $\mathcal{L}$ such that Church's Problem for conditions in $\mathcal{L}$ is solvable again in $\mathcal{L}$. Moreover, a slightly sharpened statement holds: If a condition is not solvable, then again a procedure definable in $\mathcal{L}$ exists to prohibit satisfaction of the condition. In shorter words one says that infinite games defined in $\mathcal{L}$ are determined with $\mathcal{L}$-definable winning strategies.

We shall show the result for the following logics:

1. MSO, monadic second-order logic over $(\mathbb{N}, <)$ (with free set variables, similarly for the FO-logics below),
2. FO(<), first-order logic over $(\mathbb{N}, <)$
3. FO(<)+MOD, the extension of FO(<) by modular counting quantifers,
4. FO(S), first-order logic over $(\mathbb{N}, S)$ with successor relation S,
5. strictly bounded logic, which is quantifier-free logic over $(\mathbb{N}, 0, +1)$.

The first three are treated in one proof, the last two handled separately.

We also exhibit examples of logics where the statement of the theorem fails, among them the extension of FO(S) by the quantifier $\exists^\omega$ ("there exist infinitely many") and Presburger arithmetic.

The study of the Büchi-Landweber Theorem for subsystems of MSO was started in a recent paper by Selivanov [15]. He showed that specifications given by aperiodic regular ω-languages can be solved with aperiodic transducers, which is a setting semantically equivalent to FO(<) but relying on different techniques. The present paper uses a different concept of definability and a general proof method based on Ehrenfeucht-Fraissé type equivalences of the logics under consideration. The essential proof ingredients for the logics 1. - 4. above are the following:

1. A normal form of $\mathcal{L}$-formulas in the form of Boolean combinations of formulas $\exists^\omega x \varphi(x)$, respectively $\exists x \varphi(x)$, where $\varphi(x)$ is an $\mathcal{L}$-formula bounded in x,
2. a refinement of these normal forms corresponding to parity automata, respectively weak parity automata, whose states are $\mathcal{L}$-definable equivalence types of finite words (the equivalence typically based on the undistinguishability by $\mathcal{L}$-formulas of a given quantifier rank),
3. the construction of (weak) parity games over game graphs whose vertices are essentially the mentioned equivalence classes, and the application of the positional determinacy of (weak) parity games.

These results motivate a closer study of refined uniformization problems that already have a tradition in recursion theory and descriptive set theory (however, there in the context of degrees of unsolvability). We mention some perspectives in the Conclusion.

In the subsequent Section 2 we introduce the terminology and state the main result, adding a discussion of the possible concepts of definability for solutions of Church's Problem. In Section 3 we recall the prerequisites on (weak) parity games, and in Section 4 we develop the above-mentioned normal forms. The proof of the main result follows in Section 5.

2 Preliminaries and Main Result

2.1 Church's Problem, Games, and Strategies

Church's Problem deals with sequence pairs over alphabets of the form $\{0,1\}^n$. An instance is (the formal definition of) a set S of pairs (α, β) where say $\alpha \in (\{0,1\}^{m_1})^\omega$ and $\beta \in (\{0,1\}^{m_2})^\omega$. We identify this set with the ω-language over $\{0,1\}^{m_1+m_2}$ which contains for each pair (α, β) the ω-word $\alpha(0)\hat{}\beta(0)$, $\alpha(1)\hat{}\beta(1)\dots$ (where $u\hat{}v$ denotes the concatenation of the vectors u and v).

In the original form of Church's Problem, the set S is defined in a restricted system of arithmetic. In this context, it is convenient to use the standard correspondence between a sequence $\alpha \in (\{0,1\}^n)^\omega$ and an m-tuple $(P_1, \dots, P_m)$ of predicates P_i over the natural numbers, with $i \in P_j$ iff the j-th component of $\alpha(i)$, short $(\alpha(i))_j$, is equal to 1. The underlying mathematical model is $(\mathbb{N}, <)$. We assume that the reader is acquainted with MSO logic over this structure (see [6]). We indicate first-order variables by $x, y \dots$ and monadic second-order variables by $X, Y, \dots$ Then an instance of Church's Problem is a formula $\varphi(\overline{X}, \overline{Y})$ with tuples of free set variables $\overline{X} = (X_1, \dots, X_{m_1})$ and $\overline{Y} = (Y_1, \dots, Y_{m_2})$, interpreted by tuples $\overline{P}$ and $\overline{Q}$ of predicates. For simplicity we write the ω-word associated with $\overline{P}$ as $\overline{P}(0)\overline{P}(1)\dots$, each letter being a bit vector of length m_1, similarly for $\overline{Q}$.

A solution of Church's problem for the formula $\varphi(\overline{X}, \overline{Y})$ is an operator F mapping an m_1-tuple $\overline{P}$ of predicates to an m_2-tuple $\overline{Q}$, subject to the important restriction that F should be *causal*, i.e. $\overline{Q}(n)$ should only depend on the segment $\overline{P}(0), \dots, \overline{P}(n)$ of $\overline{P}$. [3] This corresponds to the view that $\varphi(\overline{X}, \overline{Y})$ defines an infinite two-person game in which a play is built up as a sequence $\overline{P}(0), \overline{Q}(0), \overline{P}(1), \overline{Q}(1)$ etc., where the players 1 and 2 supply their choices $\overline{P}(i)$, respectively $\overline{Q}(i)$, in alternation. A strategy for Player 2 is given by a causal operator F, and a strategy for Player 1 by a *strongly causal* operator $G : \overline{Q} \mapsto \overline{P}$ (where $\overline{P}(n)$ only depends on the prefix $\overline{Q}(0), \dots, \overline{Q}(n-1)$) of $\overline{Q}$. The causal operator F is a winning strategy for Player 2 in the game defined by φ (in Church's words: a solution to the condition φ) if $\forall \overline{X} \varphi(\overline{X}, F(\overline{X}))$ holds, similarly the strongly causal operator G is a winning strategy for Player 1 if we have $\forall \overline{Y} \neg\varphi(G(\overline{Y}), \overline{Y})$. holds. All games considered in this paper are determined, i.e. either Player 1 or Player 2 has a winning strategy.

We define a causal operator $F : \overline{P} \mapsto \overline{Q}$ in terms of a word function f that assigns to any finite sequence

$$\binom{\overline{P}(0)}{\overline{Q}(0)} \dots \binom{\overline{P}(n-1)}{\overline{Q}(n-1)} \binom{\overline{P}(n)}{*}$$

a vector from $\{0,1\}^{m_2}$ (which is then taken as $\overline{Q}(n)$). By $F(\overline{P})$ we denote the sequence $\overline{Q}$ that is generated by applying f successively to $\binom{\overline{P}(0)}{*}$, to $\binom{\overline{P}(0)}{\overline{Q}(0)}\binom{\overline{P}(1)}{*}$

[3] So F is a special kind of continuous operator in the Cantor topology over infinite sequences.

etc. Similarly, a strongly causal operator G is presented as a word function g which assigns to a sequence as above, with the last column letter missing, a vector from from $\{0,1\}^{m_1}$ (which is then taken as $\overline{P}(n)$).

For the definability of an operator F in a logical system $\mathcal{L}$, we assume that $\mathcal{L}$-formulas can be interpreted in finite word models of the form displayed above. Formally, we consider structures $([0,n],<,\ (\overline{P}\cap[0,n]),\ (\overline{Q}\cap[0,n-1]),\ n)$ for causal operators $F:\overline{P}\mapsto\overline{Q}$, and $([0,n],<,\ (\overline{P}\cap[0,n-1]),\ (\overline{Q}\cap[0,n-1]),\ n)$ for strongly causal operators $G:\overline{Q}\mapsto\overline{P}$. (For the latter case we have to provide means to cover the case $n=0$, in order to fix $\overline{P}(0)$. This is done by a Boolean combination of formulas $X_i(x)$ and the statement that x has no predecessor.) We say that a function f over finite words is $\mathcal{L}$-definable if there are $\mathcal{L}$-formulas $\psi_1(\overline{X},\overline{Y},x),\ldots,\psi_{m_2}(\overline{X},\overline{Y},x)$ such that

$$([0,n],<,\ (\overline{P}\cap[0,n]),\ (\overline{Q}\cap[0,n-1]),\ n)\models\psi_j(\overline{X},\overline{Y},z)$$

iff the j-component of f applied to $\binom{\overline{P}(0)}{\overline{Q}(0)}\ldots\binom{\overline{P}(n-1)}{\overline{Q}(n-1)}\binom{\overline{P}(n)}{*}$ is equal to 1, and we call then also the associated causal operator F $\mathcal{L}$-definable. Similarly the definability of strongly causal operators is defined.

We say that an $\mathcal{L}$-defined game is *determined with $\mathcal{L}'$-definable winning strategies* if for each $\mathcal{L}$-formula $\varphi(\overline{X},\overline{Y})$, there is either an $\mathcal{L}'$-definable causal operator as winning strategy of Player 2, or an $\mathcal{L}'$-definable strongly causal operator as winning strategy for Player 1.

2.2 Fragments of MSO and Main Result

The systems MSO, FO($<$), and FO(S) are all well-known from the literature. In the first two cases, the underlying model is $(\mathbb{N},<)$, in the third case $(\mathbb{N},S)$ with the successor relation S. We use free predicate variables $X,Y,\ldots$ (for monadic predicates) in all cases; in MSO we allow also quantification over them. We write atomic formulas in the form $x=y$ (equality is included), $x<y, S(x,y)$, and $X(y)$, and use the standard connectives $\neg,\vee,\wedge,\rightarrow,\leftrightarrow$ and quantifiers $\exists,\forall$. MSO is strictly more expressive than FO($<$) (as exemplified by the formula saying "the minimum of X is even"), and FO($<$) is strictly more expressive than FO(S) (as exemplified by "there is a Y element between two X-elements").

The logic FO($<$)+MOD is obtained from FO($<$) by adjoining the quantifiers $\exists^{r,q}x$, for $q>1$ and $0\le r<q$, meaning "there is a finite number n of elements x with $n\equiv r(\text{mod } q)$". FO($<$)+MOD is a system located in expressive power strictly between FO($<$) and MSO (see [17]). Note that over $\mathbb{N}$, the quantifier allows to express $\exists^\omega x$ ("there exist infinitely many x") by negating $\exists^{r,q}$ for all $r=0,\ldots,q-1$. Denote by FO+$\exists^\omega(S)$ the extension of FO(S) by $\exists^\omega x$.

By *strictly bounded logic* mean the quantifier-free fragment of FO($0,+1$). This logic characterizes the properties of sequences that are determined by their prefixes of fixed given length (or, in other words, are both open and closed in the Cantor topology). Corresponding formulas are called "bounded specifications" in [9].

Theorem 1. (Main Theorem)
Let $\mathcal{L}$ be any of the logics MSO, FO(<), FO(<)+MOD, FO(S), or strictly bounded logic. Then each $\mathcal{L}$-definable game is determined with $\mathcal{L}$-definable winning strategies.

Theorem 2. *If $\mathcal{L}$ is FO+$\exists^\omega(S)$ or FO(S)+MOD, then there are $\mathcal{L}$-definable games that are determined, however not with $\mathcal{L}$-definable winning strategies.*

2.3 On Definability of Causal Operators

It is useful to discuss the (in general non-equivalent) options for defining causal and strictly causal operators.

An operator $F : \Sigma_1^\omega \to \Sigma_2^\omega$ is *implicitly defined* by a formula $\psi(\overline{X}, \overline{Y})$ over the structure $(\mathbb{N}, \ldots)$ if for any $\overline{P}, \overline{Q}$ we have

$$F(\overline{P}) = \overline{Q} \quad \text{iff} \quad (\mathbb{N}, \ldots) \models \psi[\overline{P}, \overline{Q}]$$

and F is said *implicitly $\mathcal{L}$ definable* iff it is defined by a formula in the logic $\mathcal{L}$. An operator F is *explicitly defined* by the formulas $\varphi_1(X_1, \ldots, X_{m_1}, x), \ldots,$ $\varphi_{m_2}(X_1, \ldots X_{m_1}, x)$ over the structure $(\mathbb{N}, \ldots)$ if for every $\bar{P} = (P_1, \ldots, P_{m_1}) \in \mathbb{P}(\mathbb{N})^{m_1}$ and $\bar{Q} = (Q_1, \ldots, Q_{m_2}) \in \mathbb{P}(\mathbb{N})^{m_2}$ the following holds:

$$\overline{Q} = F(\overline{P}) \text{ iff } (\mathbb{N}, \ldots) \models \textstyle\bigwedge_i \forall x (Q_i(x) \leftrightarrow \varphi_i(\overline{P}, x)).$$

Note that F is implicitly MSO-definable iff F is explicitly MSO-definable. However, for the fragments of MSO considered here, implicit $\mathcal{L}$-definability is (in general: strictly) more general than $\mathcal{L}$-explicit definability. As an example consider a constant operator $G : \{0,1\}^\omega \to \{0,1\}^\omega$ defined as $G(a_0, \ldots a_i \ldots) = b_0 \ldots b_i \ldots$, where $b_i = 1$ iff i is even. It is definable in $FO[<]$ implicitly but not explicitly.

Both notions are not adequate for our purpose since non causal and even non-continous functions can be definable (e.g. $F : P \mapsto Q$ with $Q = \mathbb{N}$ if P is infinite and else $Q = \emptyset$).

An operator F is *explicitly definable by bounded formulas in $\mathcal{L}$* if there are $\mathcal{L}$-formulas $\psi_i(\overline{X}, x)$ where atomic formulas $X(t)$ are only allowed for $t \leq x$ such that

$$F(\overline{P}) = \overline{Q} \quad \text{iff} \quad (\mathbb{N}, \ldots) \models \bigwedge_i \forall x (Q_i(x) \leftrightarrow \psi_i(\overline{P}, x))$$

Let $\mathcal{L}$ be one of the following logics MSO, $FO[<]$ and for $FO[<]$+MOD. Clearly, if an operator is $\mathcal{L}$-definable by bounded formulas, then it is causal.

However, this notion is insufficient for weak logics such as FO(<), due to a lack of reference to previous values; note that in MSO a sequence of values $Q(y)$ for $y < x$ is accessible by an auxiliary second-order quantification. An option to implement this reference is to use the full play prefix $P(0)Q(0)P(1)Q(1)\ldots$ $P(n{-}1)Q(n{-}1)P(n)$ when defining $Q(n)$. But this does not allow (in FO(<) and FO(S)) to determine the origin of, say, a single bit 1 (assuming 0 elsewhere), namely whether this bit belongs to the $P(i)$ or to the $Q(i)$. Thus we use the vector representation for the pairs $(\overline{P}(i), \overline{Q}(i))$.

Finally, we remark that the notion of bounded formula is meaningful only when the order relation $<$ (or $\leq$) is in the signature. In order to cover also the successor logic FO(S), we pass to finite prefixes of infinite sequences as the models of formulas that define strategies, shifting boundedness from the syntactic to the semantic level. We thus arrive at the concept of definability for causal and strongly clausal operators as used above in Theorem 1.

3 Preliminaries on Games and Automata

3.1 Muller and Parity Games and Their Weak Versions

A directed bipartite graph $G = (V_1, V_2, E)$ is called a *game arena* if the outdegree of every vertex is at least one. We assume in this paper that $V := V_1 \cup V_2$ is finite. If G is an arena, a game on G is defined by an initial node $v_{init} \in V_1$ and a winning condition (by convention for Player 2). A play over G is an infinite sequence $\rho \in V^\omega$ starting in v_{init}, built up by two agents called player 1 and 2 that choose edges in alternation (player i from vertices in V_i). By $\mathrm{Inf}(\rho)$, respectively $\mathrm{Occ}(\rho)$ we denote the set of vertices from V which occur infinitely often, respectively just occur, in ρ. A winning condition decides who wins a play; we declare Player 2 to be the winner iff it is satisfied. (Here we follow the convention above that the "good" player is the second, having to react move by move to Player 1.) First we consider two winning conditions, called Muller condition, respectively weak Muller condition (the latter is also called Staiger-Wagner-condition in the literature). In both cases, the condition is specified by a family $\mathcal{F}$ of subsets of V. A play ρ satisfies the Muller (resp. weak Muller) condition $\mathcal{F}$ if $\mathrm{Inf}(\rho) \in \mathcal{F}$ (resp. if $\mathrm{Occ}(\rho) \in \mathcal{F}$).

As usual, a strategy f for Player 1 (Player 2) is a function which assigns to every path of odd (positive even) length a node adjacent to the last node of the path. (We assume that the inital vertex is given, and that player 1 starts from there.) A play $v_{init}v_1v_2\ldots$ is played according to a strategy f_1 of Player 1 (strategy f_2 of Player 2) if for every prefix $\pi = v_{init}v_1\ldots v_n$ of odd (even) length we have $v_{n+1} = f_1(\pi)$ (respectively, $v_{n+1} = f_2(\pi)$). A strategy is winning for Player 2 (respectively, for Player 1) if all the plays played according to this strategy satisfy the winning condition under consideration. A strategy is *memoryless* if it depends only on the respective last node in the path.

In a *parity game* (resp. *weak parity game*), the winning condition refers to a coloring $c : V_1 \cup V_2 \to \{0, 1, \ldots m\}$ of the game graph. For a play $\rho = v_0v_1\ldots$ let $C_\omega(\rho)$, resp. $C(\rho)$, be the maximal color occurring infinitely often, resp. occurring at all, in the sequence $c(v_0)c(v_1)\ldots$. A play ρ is won according to the parity condition, resp. weak parity condition, if $C_\omega(\rho)$, resp. $C(\rho)$, is even.

The following theorem, due to Emerson/Jutla and Mostowski, is fundamental (see [6, 12] and e.g. [21] for the weak case):

Theorem 3 (Memoryless Determinacy for (Weak) Parity Games)
In a parity game, one of the two players has a winning strategy which moreover is memoryless. The same statement holds for weak parity games.

3.2 From (Weak) Muller to (Weak) Parity Conditions

There are well-known reductions of Muller games and weak Muller games to parity games, respectively weak parity games. We recall these constructions here as a background to the next section. The idea is to transform an infinite sequence (e.g., an infinite play) ρ over the set V into a new sequence ρ' over a larger alphabet, which results from V by adding a "memory component". A coloring is associated with each of the new letters, making use of the state-set collection $\mathcal{F}$ that defines the (weak) Muller condition under consideration. The aim is to obtain the following equivalence: The sequence ρ satisfies the (weak) Muller condition iff ρ' satisfies the (weak) parity condition.

First we treat the case of weak Muller conditions, say for a collection $\mathcal{F} \subseteq 2^V$. We associate with each sequence $\rho = v_0 v_1 \ldots$ over V a sequence over $V \times 2^V$, where the first components give a copy of ρ and a second component $P \subseteq V$ indicates the set of previously visited V-elements. We thus speak of the data structure *appearence record*, short AR. So the sequence ρ' starts with $(v_0, \emptyset)$, and for a step from v to v' in ρ we pass in ρ' from (v, P) to $(v', P \cup \{v\})$. The sequence ρ' will have, from some point onwards, a constant second component, which equals $\mathrm{Occ}(\rho)$. If we attach to (v, P) the color $2|P|$ when $P \in \mathcal{F}$ and $2|P| - 1$ when $p \notin \mathcal{F}$, then we have that $\mathrm{Occ}(\rho) \in \mathcal{F}$ iff $C(\rho)$ is even (which means that the weak parity condition is satisfied).

A similar reduction is known for the Muller condition (see [19]). Consider again $\mathcal{F} \subseteq 2^V$, where $V = \{v_1, \ldots, v_n\}$. Now not only the visited V-elements but also the order of their visits is recorded (*latest appearance record*, short LAR [7]). Let $\rho = v_0 v_1 \ldots v_j \ldots$ be a sequence over V. The associated latest appearance record at time point j is a sequence $(v_j, v_{i_1}, \ldots, v_{i_k})$ where $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is the appearance record at j, i.e., the set of states visited before j. We list them in the order of latest appearance before v_j (most recently visited vertices listed first): Every state from S appears exactly once in the sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$, and we have a sequence $j > j_1 > j_2 > \ldots > j_k$ such that: (1) v_{i_m} appears at j_m in ρ ($m = 1, \ldots k$); (2) There is no occurrence of v_{i_m} between positions j_m and j in ρ ($m = 1, \ldots k$).

If in the LAR $(v, v_{i_1}, \ldots v_{i_m})$, v occurs as entry v_{i_h}, we call h the "index" of the LAR. Using the index we equip a LAR with a color as follows. Assign to $(v, v_{i_1}, \ldots v_{i_m})$ of index h the color $2h$ if $\{v_{i_1}, \ldots, v_{i_h}\} \in \mathcal{F}$ and otherwise $2h - 1$. Then the sequence ρ satisfies the Muller condition w.r.t. $\mathcal{F}$ iff the induced sequence ρ' satisfies the parity condition with respect to the defined coloring.

Two observations are of interest in the sequel:

1. The description of the LAR structure requires the order relation of the underlying model, while this is irrelevant for the AR structure.

2. Together with the positional determinacy of (weak) parity games, the reductions yield finite-state automata that execute winning strategies in (weak) Muller games; the state set (consisting of LAR's, respectively AR's) and the transition function of a strategy automaton depend only on the game graph, whereas for the output function the winning condition is also used.

4 Normal Forms

As a tool to analyze logical formulas we recall well-known equivalences between models based on indistinguishability by formulas up to some quantifier depth k. We call these equivalence classes "k-types". The models under consideration are expansions of finite orderings $(A, <)$ or of $(\mathbb{N}, <)$ by tuples $\overline{P}$ of unary predicates. For m-tuples $\overline{P}$ we speak of m-chains.

The logics $\mathcal{L}$ to be considered below are MSO, FO($<$), FO($<$)+MOD, and FO(S).

4.1 Types

Two m-chains M, M' are k-equivalent for $\mathcal{L}$ (written: $M \equiv_k^{\mathcal{L}} M'$) if $M \models \varphi \Leftrightarrow M' \models \varphi$ for every $\mathcal{L}$-formula $\varphi(\overline{X})$ of quantifier depth k. This is an equivalence relation between m-chains; its equivalence classes are called k-types for $\mathcal{L}$ (and for the considered signature and m unary predicates). We denote the (finite) set of k-types by H_k, usually suppressing an index for the logic under consideration. (In the case of FO[$<$]+MOD, we assume that also a maximal divisor q is fixed in advance.) If a formula φ is true in every model of type t we say that "t implies φ".

Let us list some fundamental and well-known properties of k-types for any of the mentioned logics $\mathcal{L}$ above; we suppress the reference to L for simplicity of notation.

Proposition 4. *1. For every m and k there are only finitely many k-types of m-chains.*
2. For each k-type t there is a "characteristic formula" which defines t (i.e., is satisfied by an m-chain iff it belongs to t). For given k and m, a finite list of characteristic formulas for all the possible k-types can be computed.
3. Each formula $\varphi(\overline{X})$ is equivalent to a (finite) disjunction of characteristic formulas; moreover, this disjunction can be computed from φ.

The proofs of these facts can be found in several sources, we mention [8, 16, 20] for MSO and FO($<$), [17] for FO(S) and FO($<$)+MOD.

Given m-labelled chains M_0, M_1 we write $M_0 + M_1$ for their concatenation (ordered sum). In our context, M_0 will always be finite and M_1 finite or of order type ω. Similarly, if for $i \geq 0$ the chain M_i is finite, the model $\Sigma_{i \in \mathbb{N}} M_i$ is obtained by the concatenation of all M_i in the order given by the index structure $(\mathbb{N}, <)$.

We need the following composition theorem on ordered sums ([16]):

Proposition 5. (Composition Theorem)
Let $\mathcal{L}$ be any of the logics considered above.

(a) *The k-types of m-chains M_0, M_1 for $\mathcal{L}$ determine the k-type of the ordered sum $M_0 + M_1$ for $\mathcal{L}$, which moreover can be computed from the k-types of M_0, M_1.*
(b) *If the m-chains $M_0, M_1, \ldots$ all have the same k-type for $\mathcal{L}$, then this k-type determines the k-type of $\Sigma_{i \in \mathbb{N}} M_i$, which moreover can be computed from the k-type of M_0.*

Part (a) of the theorem justifies the notation $s+t$ for the k-type of an m-chain which is the sum of two m-chains of k-types s and t, respectively.

4.2 Strong Normal Forms

In this subsection we deal with the logics MSO, FO(<)+MOD, and FO(<), and we denote here by $\mathcal{L}$ any of these three logics.

A formula $\psi(x)$ with at most one free individual variable x is (syntactically) *bounded* if all its first-order quantifiers are of the form $\exists^{\leq x} y \ldots$ (short for $\exists y(y \leq x \wedge \ldots)$ and $\forall^{\leq x} y \ldots$ (short for $\forall y(y \leq x \rightarrow \ldots)$).

We use a normal form proved in [18] for MSO and FO(<) which easily extends to FO(<)+MOD. For all logics that satisfy the Composition Theorem, it provides a first-order description of the fact (for given k) that the infinite m-chain under consideration can be cut into a sequence $w_0, w_1, w_2, \ldots$ such that all w_i for $i > 0$ share the same k-type t. (The fact that such a decomposition exists is clear from Ramsey's Theorem; the obvious formalization, however, uses a second-order quantifier.)

Proposition 6 ([18]). *Every $\mathcal{L}$-formula $\varphi(\overline{Z})$ is equivalent (over the class of m-chains with domain $\mathbb{N}$) to a formula in* bounded normal form, *more specifically of the form*

$$\bigvee_{i=1}^{n} (\exists^{\omega} z \psi_i(\overline{Z}, z) \ \wedge \neg \exists^{\omega} z \psi'_i(\overline{Z}, z))$$

where the ψ_i, ψ'_i are bounded.

In a next step we sharpen this normal form to a "parity normal form".

Formulas in parity normal form involve a coloring of a certain finite set of bounded formulas $\Delta = \{\varphi_1(\overline{Z}, x), \ldots, \varphi_{n+1}(\overline{Z}, x)\}$ such that an m-chain can satisfy at most one of them. We denote the color of φ_i by $col(\varphi_i)$ and call Δ_j the formulas from Δ of color $\leq j$.

Lemma 7. (Parity Normal Form)
Every $\mathcal{L}$-formula $\varphi(\overline{Z})$ is equivalent (over the class of m-chains with domain $\mathbb{N}$) to a formula in parity normal form:

$$\bigvee_{i=0}^{n/2} (\bigvee_{\varphi \in \Delta_{2i}} \exists^{\omega} x \ \varphi(\overline{Z}, x) \quad \wedge \bigwedge_{\varphi \in \Delta_{2i+1}} \neg \exists^{\omega} x \ \varphi(\overline{Z}, x)))$$

where Δ is a finite set of bounded formulas, such that an ω-chain satisfies at most one of them, and $\Delta_0 \subseteq \ldots \subseteq \Delta_{n+1} = \Delta$. (This formula will be denoted $Parity_\varphi(\Delta, col)$.)

Proof. The proof follows the mentioned transformation from Muller automata to parity automata. The only important observation is that the latest appearance record LAR can be formalized in FO(<).

We let $\varphi(\overline{Z})$ be a formula and let ψ_i, ψ'_i be formulas as in Proposition 6. Let k be the maximal quantifier depth of the formulas $\psi_i(\overline{Z}, x), \psi'_i(\overline{Z}, x)$ of the above bounded normal form of $\varphi(\overline{Z})$. Again, let H_k be the set of k-types over $\overline{Z}$.

Let us define the following Muller acceptance condition $\mathcal{F} \subseteq \mathbb{P}(H_k)$: We include a subset R of H_k in the set $\mathcal{F}$ iff for some j, R contains some type implying ψ_j but contains no type implying ψ'_j. Note that a structure $M = (\mathbb{N}, \ldots)$ satisfies φ iff the set of k-types that are satisfiable cofinally often on the initial substructures of M is in $\mathcal{F}$.

For every latest appearance record $lar = (t, t_{i_1}, \ldots, t_{i_m})$ over H_k, we can write a formula $\varphi_{lar}(\overline{Z}, x)$ such that $(\mathbb{N}, <) \models \varphi_{lar}(\overline{P}, j)$ iff lar is the latest appearance record of the sequence $t_1 \ldots t_j$, where t_i is the k-type of of the submodel of $(\mathbb{N}, <, \overline{P})$ over the interval $[0, i]$. For every k-type t one can write in these logics a formula expressing that the k-type of an interval $[0, x]$ is t (We denote this formula by $T^k(x) = t$, suppressing $\overline{Z}$). Then φ_{lar} can be easily expressed as explained in Subsection 3.2. For example, if $lar = (t, t_{i_1}, t, t_{i_2})$, then φ_{lar} is the conjunction of the following formulas:

$$T^k(x) = t \wedge \forall y < x(T^k(y) = t \vee T^k(y) = t_{i_1} \vee T^k(y) = t_{i_2})$$

$$\begin{aligned}\exists y_1 y_2 y(y_2 < y < y_1 < x) \wedge & (T^k(y_1) = t_{i_1} \wedge T^k(y_2) = t_{i_2} \wedge T^k(y) = t = T^k(x) \\ & \wedge \forall z\big((x > z > y) \rightarrow \neg T^k(z) = t\big) \\ & \wedge \forall z\big((x > z > y_1) \rightarrow \neg T^k(z) = t_{i_1}\big) \\ & \wedge \forall z\big((x > z > y_2) \rightarrow \neg T^k(z) = t_{i_2}\big)\end{aligned}$$

Now let $\mathcal{F}$ be the acceptance Muller conditions over H_k which corresponds to presentation of φ in bounded normal form. We transform Muller conditions into a coloring of LAR over H_k exactly like explained in Section 3.2, and for $lar \in LAR$ will assign to φ_{lar} the color of lar. Let Δ be the set of formulas $\{\varphi_{lar} : lar$ is a latest appearance record over $H_k\}$. It is easy to verify that φ is equivalent to the formula $Parity_\varphi(\Delta, col)$.

4.3 Weak Normal Forms

For the logic FO(S), we consider models $M = (\mathbb{N}, S, \overline{P})$ and $M = ([0, n], S, \overline{P})$ with an m-tuple $\overline{P}$. We first recall the model theoretic analysis of FO(S) which relies on the first-order model theory of finite graphs (due to Hanf, see e.g. [5]).

By the r-sphere around the element x we mean the submodel, pointed at x, consisting of x with its next k neighbours to the left and to the right (as far as these neighbors exist). We call a sphere right-complete (left-complete) if these k elements exist to the right (left) of x, and complete if both properties apply. By

σ we denote an isomorphism type of an r-sphere; the set of all possible r-sphere isomorphism types is denoted S_r.

A (r, K)-type of a model M is given, for each $\sigma \in S_r$, by the numbers n_σ of occurrences of σ counted up to threshold K. So it is a vector $(n_\sigma)_{\sigma \in S_r}$ of values in $[0, K+1]$ and defined by a conjunction of FO(S)-formulas "there are precisely k elements x with: "r-sphere type of $x = \sigma$" where $k < K$, and "there are $K+1$ elements x with "r-sphere type of $x = \sigma$".

We need the following known facts which give a rather direct description of k-types for FO(S):

Lemma 8. (Hanf, see [5])
For each k, there are r, K such that each (r, K)-type implies a fixed k-type (so (r, K)-types refine k-ypes), or in other words: Truth of a FO(S)-formula of quantifier depth k over a model M as considered here is determined by the (r, K)-type of M.

Now we consider models with domain $\mathbb{N}$. Then only right-complete spheres are relevant. We use this fact to introduce a restricted version of type for the finite prefixes (ignoring the right-incomplete spheres of prefixes), which induces a monotonicity property when we let the prefixes increase. For the initial segment M_s of a model $M = (\mathbb{N}, S, \overline{P})$ up to number s we denote by $\pi(s) = (n_\sigma)_\sigma$ the vector of natural numbers in $[0, K+1]$ which lists, in some fixed order of the r-sphere types σ that are right-complete, the numbers of their occurrences in M_s counted up to threshold value K (again representing any number $> K$ by $K+1$). Call $\pi(s)$ the "(r, K)-profile" of M_s, and π the (r, K)-profile of the whole structure M. From the lemma it follows (under the given conventions for the parameters k, r, K) that the (r, K)-profile π determines truth of formulas $\varphi(\overline{Z})$ in M up to quantifier depth k.

When s increases, the profiles $\pi(s)$ can only increase as well (componentwise). At some point s_0, the value $\pi(s_0) = (n_\sigma^{s_0})_\sigma$ is equal to the M-profile π and stays constant. Let us write $\pi' > \pi$ if for all components the π'-value is $\geq$ the corresponding π-value, and for some component we have a strict inequality $>$. We can write down FO(S)-formulas φ_τ, $\varphi_{>\tau}$ expressing in a segment model M_s that its (r, K)-profile is τ, respectively $> \tau$.

We obtain the following "normal form" for FO(S)-formulas; note that we write it down on the semantical level since bounded formulas are not available in FO(S).

Lemma 9. ("Weak Normal Form")
Let $\varphi(\overline{Z})$ be a FO(S)-formula. Then for each model $M = (\mathbb{N}, S, \overline{P})$, we have $M \models \varphi(\overline{Z})$ iff

$$\bigvee_{\tau \in \Pi(\varphi)} \big(\exists s\ M_s \models \varphi_\tau\ \wedge\ \neg\exists s\ M_s \models \varphi_{>\tau}\big)$$

where $\Pi(\varphi)$ is the set of (r, K)-profiles that imply φ.

A small further step gives us a parity normal form. For this we consider an extension of the partial order of (r, K)-profiles to a linear order, giving each profile an index h. We associate now colors with the formulas $\varphi_\tau, \varphi_{>\tau}$ above, by giving a profile of index h the color $2h$ if it belongs to $\Pi(\varphi)$, otherwise $2h-1$. Assume the colors range from 0 to $n+1$. Then we obtain the following, using again the notation of Δ_j for the formulas of color $\leq j$:

Lemma 10. ("Weak Parity Normal Form")
Let $\varphi(\overline{Z})$ be a FO(S)-formula. For each model $M = (\mathbb{N}, S, \overline{P})$, $M \models \varphi(\overline{Z})$ iff

$$\bigvee_{i=0}^{n/2} \Big(\bigvee_{\varphi_\tau \in \Delta_{2i}} \exists s\ M_s \models \varphi_\tau \quad \wedge \bigwedge_{\varphi_\sigma \in \Delta_{2i+1}} \neg \exists s\ M_s \models \varphi_\sigma \Big)$$

5 Proof of Theorems 1 and 2

5.1 Logics with Strong or Weak Normal Form

Let $\mathcal{L}$ be any of the logics MSO, FO(<), FO(<)+MOD. In the previous section we have shown that each $\mathcal{L}$-formula $\varphi(\overline{Z})$ can be transformed into an equivalent parity automaton $\mathcal{A}_\varphi$ whose states are k-types for the logic $\mathcal{L}$, for suitable k. (In order to include the empty model as initial state, the state space of $\mathcal{A}_\varphi$ is $H_k \cup \{\epsilon\}$.) After scanning an initial segment $\overline{P}(0)\ldots\overline{P}(n)$, the automaton assumes just the k-type (for $\mathcal{L}$) of the model $([0,n], <, (\overline{P} \cap [0,n]))$. By construction (and by the properties of k-types), each state is $\mathcal{L}$-definable.

This transformation of a specification (game definition) $\varphi(\overline{Z})$ into an automaton is independent of the game theoretical context. Now we emphasize this aspect again. The m-tuple $\overline{Z}$ is split into two blocks $\overline{X}, \overline{Y}$ of length m_1, m_2, respectively, the specification reads $\varphi(\overline{X}, \overline{Y})$, and predicates $\overline{P}, \overline{Q}$ used for the interpretation of $\overline{X}, \overline{Y}$ are built up step by step in alternation.

Following this splitted construction, we introduce a game graph, where the vertices from $H_k \cup \{\epsilon\}$ are accompanied by extra vertices in $(H_k \cup \{\epsilon\}) \times \{0,1\}^{m_1}$, which serve to represent the "intermediate steps" reached by Player 1 after his choice of a m_1-tuple $\overline{P}(n)$.

Formally, we define the game graph $G_\varphi = (V_1, V_2, E)$ by

- $V_1 = H_k \cup \{\epsilon\}$, $V_2 = V_1 \times \{0,1\}^{m_1}$
- the edge set E with an edge from $t \in V_1$ to $(t, \overline{a})$ for each $t \in V_1$ and each $\overline{a} \in \{0,1\}^{m_1}$, and an edge from each $(t, \overline{a})$ to the k-type $t + (\overline{a}, \overline{b})$ for each $\overline{b} \in \{0,1\}^{m_2}$. (Recall that $t + (\overline{a}, \overline{b})$ is the k-type of a model which results from a model of type t by concatenating the m-tuple $(\overline{a}, \overline{b})$.)

In order to obtain a parity game, we have to define a coloring c. For this, we associate to both t and each $(t, \overline{a})$ the same color as given for t in the automaton $\mathcal{A}_\varphi$. Then it is obvious that a sequence $(\overline{P}, \overline{Q})$ satisfies φ iff Player 2 wins the parity game over G_φ with the coloring c. Assume that Player 2 wins. Then we can fix a winning strategy by choosing one m_2-tuple $\overline{b}$ for each vertex $(t, \overline{a})$ in V_2. Denote the i-th component of this vector $\overline{b}$ by $b_i(t, \overline{a})$. We show that this strategy is $\mathcal{L}$-definable. For this, recall that we can express by an $\mathcal{L}$-formula $\psi_t(\overline{X}, \overline{Y}, x)$

that "the k-type of $([0, x-1], (\overline{X} \cap [0, x-1]), (\overline{Y} \cap [0, x-1], x)$ is t". We define the winning strategy by the following $\mathcal{L}$-formulas $\psi_i(\overline{X}, \overline{Y}, x)$:

$$\bigvee_{(t,\overline{a}) \in V_2} (\psi_t(\overline{X}, \overline{Y}, x) \wedge \overline{X}(x) = \overline{a} \wedge \text{"}b_i(t, \overline{a}) = 1\text{"})$$

Here $\overline{X}(x) = \overline{a}$ stands for $\bigwedge_j [\neg] X_j(x)$ with negations inserted for those j where $a_j = 0$, and $b_i(t, \overline{a}) = 1$ for "true" or "false" depending on the value of b_i. The proof for definability of a winning strategy for Player 1 works similarly.

For the logic FO(S), we proceed in exact analogy, invoking the weak parity normal form, constructing a weak parity game consisting of FO(S)-definable states, and using positional determinacy of weak parity games. Since compositionality of types is needed in the definition of the game graph, one uses (r, K)-types as vertices, but the induced (r, K)-profiles for the winning condition.

5.2 Strictly Bounded Logic

For a specification $\varphi(\overline{X}, \overline{Y})$ of strictly bounded logic (the quantifier-free fragment of FO$(0, +1)$) with an m_1-tuple $\overline{X}$ and a m_2-tuple $\overline{Y}$, let k be the maximal nesting of the function symbol $+1$ in φ. It is easy to show that satisfaction of φ in $(\mathbb{N}, 0, +1, \overline{P}, \overline{Q})$ only depends on the prefix of $(\overline{P}, \overline{Q})$ up to position k. Collect the finite set L_0 of these prefixes (of length $k+1$) such that all their extensions to ω-sequnces satisfy φ. We consider the game graph with vertices $w \in \{0, 1\}^{m_1+m_2}$ of length $\leq k+1$ (for Player 1) and $(w, \overline{a})$ for these w and $\overline{a} \in \{0, 1\}^{m_1}$ (for Player 2). The standard attractor construction (see [6]) yields a positional winning strategy for either of the two players which is clearly definable in strictly bounded logic.

5.3 Proof of Theorem 2

We consider a game due to Dziembowski, Jurdziński, and Walukiewicz [4]. For better readability we use the alphabet $\{a, b, c\}$ for X and $\{0, 1, 2\}$ for Y. Let $\varphi(X, Y)$ be the following condition: "If a, b occur infinitely often in X, then 2 occurs infinitely often in Y; if only one of a, b occurs infinitely often, then 1, but not 2, occurs infinitely often in Y; otherwise Y is ultimately 0". This condition is expressible in FO+$\exists^\omega(S)$ (even without S).

The game is solvable by means of the LAR over $\{a, b, c\}$. The output is 0 if c is the current X-value, it is 1 if a is the current value with b occurring after a in the current LAR-list, and otherwise 2 (dually for letter b).

Assume that a winning strategy is definable in FO+$\exists^\omega(S)$. Since the underlying models are finite words, it is easy to transform the definition into an equivalent FO(S)-definition. Choose (r, K) as in the weak normal form theorem and consider the word $u = (c^{2r} a c^{2r} b)^K c^{2r}$ over $\{a, b, c\}$. Now we apply a case distinction concerning the output values of the strategy for words in uac^* after u. If the maximum is 0 or 2, we obtain a contradiction to the assumption that the strategy wins, by considering $P = u(ac^{2r})^\omega$; namely, the strategy will yield 0, respectively 2, as the maximal output repeated infinitely often but should do this with value 1. Similarly, one argues for the case of words in ubc^*. It remains to consider the situation that for both cases the maximum output is 1. Then we obtain a contradiction for $u(ac^{2r}bc^{2r})^\omega$; the strategy yields 1 as maximal output repeated infinitely often but should produce value 2.

The proof for FO(S)+MOD works similarly (since again the quantifier $\exists^\omega$ is implicitly present), however with a more involved case distinction which is omitted here.

6 Discussion and Perspectives

Based on a natural concept of logical definability of winning strategies in infinite games, we exhibited several fragments $\mathcal{L}$ of MSO logic such that the $\mathcal{L}$-definable games are determined with $\mathcal{L}$-definable winning strategies.

Let us add remarks on possible extensions of these results.

Our formulation of Church's Problem can be viewed as a task to transform ω-languages (specifications) to tuples of standard languages (defining the output functions, essentially by descriptions of mutually disjoint languages of play prefixes for the different output letters). This study can be pursued in language theory, and further types of properties can be considered, like "locally testable" or "piecewise testable" (see [13]). These properties are not captured by logics but can be analyzed in a very similar way by approriate equivalence relations. We leave a more detailed treatment (which involves some extension of the method of the present work) to a future paper.

Turning to extensions of MSO over $(\mathbb{N}, <)$, we first mention that by an easy word-for-word translation of the proofs, we can cover the case of structures $(\mathbb{N}, <, \mathbb{P}_1, \ldots, \mathbb{P}_n)$ with fixed subsets $\mathbb{P}_i$ as "parameters". The only difference occurs in the effectiveness of constructing the definition of winning strategies (namely, when the $\mathcal{L}$-theory of $(\mathbb{N}, <, \mathbb{P}_1, \ldots, \mathbb{P}_n)$ is undecidable). Another interesting structure is $(\mathbb{N}, <, f)$ with the "flip function" f; it assigns to each number n the number obtained from n by flipping in the binary expansion of n the highest 1-bit to 0. Based on the compositional decidability proof of the MSO-theory of $(\mathbb{N}, <, f)$ ([10]), we conjecture that Theorem 1 can be extended to MSO-specifications over $(\mathbb{N}, <, f)$.

Both kinds of extensions are inessential in the sense that they are (respectively, seem) accessible by an adaption of the method of this paper. A decidable theory which falls outside this frame is, for example, Presburger arithmetic, the first-order theory of addition over $\mathbb{N}$. Here Theorem 1 fails: [4] It is easy to write down in Presburger arithmetic a formula $\varphi(Y_0)$ which says that Y_0 is the set Squ of squares (use the fact that the distances of successive squares increase by 2).

[4] The short abstract [2] of Büchi, Elgot, and Wright (without a corresponding paper) claims that specifications in Presburger arithmetic do not have, in general, MSO-definable solutions; the reference to [14] given in [2] seems to point to a similar argument as sketched here.

The strategy to generate Squ is also Presburger definable. Invoking the fact that multiplication is FO-definable in $(\mathbb{N}, +, \text{Squ})$ ([14]), the FO(+)-specifications are the arithmetical relations in $(\overline{X}, Y_0, \overline{Y})$ where $Y_0 = \text{Squ}$. On the other hand, it is known ([11]) that there are specifications $\exists^{\omega} x R(\overline{X}, \overline{Y}, x)$ even with recursive R which (are determined but) do not allow an arithmetical winning strategy. This leads us to the question: Are there natural logics $\mathcal{L}$ which are not covered by the proof method of this paper and still satisfy Theorem 1?

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